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The formal theory of monads II ☆

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Abstract

We give an explicit description of the free completion $\mathsf{EM}(\mathscr{K})$ of a 2-category $\mathscr{K}$ under the Eilenberg–Moore construction, and show that this has the same underlying category as the 2-category $\mathsf{Mnd}(\mathscr{K})$ of monads in $\mathscr{K}$. We then demonstrate that much of the formal theory of monads can be deduced using only the universal property of this completion, provided that one is willing to work with $\mathsf{EM}(\mathscr{K})$ as the 2-category of monads rather than $\mathsf{Mnd}(\mathscr{K})$. We also introduce the *wreaths* in $\mathscr{K}$; these are the objects of $\mathsf{EM}(\mathsf{EM}(\mathscr{K}))$, and are to be thought of as generalized distributive laws. We study these wreaths, and give examples to show how they arise in a variety of contexts.

MSC: 18C20; 18C15; 18A35; 18D05; 18G15; 16W30

0. Introduction

A monad consists of a category A, a functor $t : A \to A$, and natural transformations $\mu : t^2 \to t$ and $\eta : 1_A \to t$ satisfying three equations, as expressed by the commutative diagrams

$$
\begin{array}{ccc}
t^3 & \xrightarrow{t\mu} & t^2 \\
{\scriptstyle \mu t}\downarrow & & \downarrow{\scriptstyle \mu} \\
t^2 & \xrightarrow[\mu]{} & t
\end{array}
\qquad\qquad
\begin{array}{ccccc}
t & \xrightarrow{t\eta} & t^2 & \xleftarrow{\eta t} & t \\
 & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\
 & & t. & &
\end{array}
$$

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Jean Bénabou [2] realized that this definition can be made in an arbitrary 2-category $\mathscr{K}$: rather than the category A one has an object A of $\mathscr{K}$, rather than the functor t one has an arrow $t: A \to A$ of the 2-category, and rather than the natural transformations η and μ one has 2-cells in the 2-category; the equations still make perfectly good sense. Then monads in the usual sense are just monads in the 2-category Cat. In fact, Bénabou used not 2-categories, but the more general *bicategories*; in this context the equations have to be modified slightly using the structural isomorphisms $(tt)t \cong t(tt)$ and $t1_A \cong t \cong 1_A t$. Also, he observed that a monad in the bicategory $\mathscr{B}$ was precisely a *morphism of bicategories* (also called a lax functor) from 1 to $\mathscr{B}$. The observation that monads can be defined in an arbitrary 2-category is what one might call *the formal definition of monads*, while *the formal theory of monads* consists in developing the usual elements of the theory of monads—such as the Kleisli and Eilenberg–Moore categories of a monad, and distributive laws between monads—in the context of monads in a 2-category $\mathscr{K}$. This program was begun by the second author in [21]; we shall now outline the most important aspects. The first step of this program was the idea that the monads in a 2-category $\mathscr{K}$ are themselves the objects of a 2-category, called $\mathsf{Mnd}(\mathscr{K})$; this is the basic organizational tool for the formal theory of monads as developed in [21]. We shall write (A,t) and (B,s) for typical monads in $\mathscr{K}$, leaving the multiplication and unit understood; these will almost always be denoted by μ and η. The 1-cells in $\mathsf{Mnd}(\mathscr{K})$ are the *monad morphisms*: a monad morphism from (A,t) to (B,s) consists of a 1-cell $f: A \to B$ and a 2-cell $\phi: sf \to ft$ satisfying two equations:

$$\begin{array}{ccccc} ssf & \xrightarrow{s\phi} & sft & \xrightarrow{\phi t} & ftt \\ {\scriptstyle \mu f}\downarrow & & & & \downarrow{\scriptstyle f\mu} \\ sf & & \xrightarrow[\phi]{} & & ft \end{array} \qquad \begin{array}{ccc} & f & \\ {\scriptstyle \eta f}\swarrow & & \searrow{\scriptstyle f\eta} \\ sf & \xrightarrow[\phi]{} & ft \end{array}$$

A 2-cell in $\mathsf{Mnd}(\mathscr{K})$ from (f,ϕ) to (g,ψ) is a *monad transformation*, that is, a 2-cell $\rho: f \to g$ in $\mathscr{K}$ satisfying the single condition

$$\begin{array}{ccc} sf & \xrightarrow{\phi} & ft \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho t} \\ sg & \xrightarrow[\psi]{} & gt. \end{array}$$

The evident compositions and identities (see [21]) make this into a 2-category $\mathsf{Mnd}(\mathscr{K})$. There is a fully faithful 2-functor $\mathsf{Id}: \mathscr{K} \to \mathsf{Mnd}(\mathscr{K})$ which sends an object A to the identity monad $(A,1)$ on A for which the multiplication and unit are both identity 2-cells.

For a monad (A,t) in the usual sense, an algebra for t consists of an object a of A, equipped with a map $ta \to a$ satisfying certain conditions. Now objects of a general 2-category do not themselves have objects, so in our formal theory we consider arbitrary arrows $a: X \to A$ with codomain A, and actions $ta \to a$ satisfying the usual conditions, and these become our algebras. In fact, given a monad (A,t) in $\mathscr{K}$ and an object X of $\mathscr{K}$ one may consider the hom-category $\mathscr{K}(X,A)$, and the induced monad

$\mathscr{K}(X,t)$ thereon, whose endofunctor parts sends an object $a:X \to A$ of $\mathscr{K}(X,A)$ to $ta:X \to A$; and an algebra for $\mathscr{K}(X,t)$ is thought of as a "generalized t-algebra with domain X".

For any object X of $\mathscr{K}$ one may form the Eilenberg–Moore category $\mathscr{K}(X,A)^{\mathscr{K}(X,t)}$ of the monad $\mathscr{K}(X,t)$, and this defines the object-part of a 2-functor $\mathscr{K}^{\mathrm{op}} \to \mathsf{Cat}: X \mapsto \mathscr{K}(X,A)^{\mathscr{K}(X,t)}$. If this 2-functor is representable, we write A^t for the representing object, and call it the *Eilenberg–Moore object* of the monad (A,t). From [21] we have:

Proposition 0.1. *To give a right adjoint to* $\mathsf{Id}:\mathscr{K} \to \mathsf{Mnd}(\mathscr{K})$ *is precisely to give a choice, for each monad in* $\mathscr{K}$, *of an Eilenberg–Moore-object of the monad. In particular, there exists a right adjoint to* Id *if and only if* $\mathscr{K}$ *has Eilenberg–Moore objects.*

Eilenberg–Moore objects in Cat are just the usual Eilenberg–Moore categories, while similarly Eilenberg–Moore objects in the 2-category $\mathscr{V}$-Cat of $\mathscr{V}$-categories, $\mathscr{V}$-functors, and $\mathscr{V}$-natural transformations are just the usual (enriched) Eilenberg–Moore $\mathscr{V}$-categories. The 2-category $\mathsf{Cat}^{\mathrm{op}}$ is obtained by reversing the 1-cells but not the 2-cells of Cat, and a monad in $\mathsf{Cat}^{\mathrm{op}}$ is the same thing as a monad in Cat, but Eilenberg–Moore objects in $\mathsf{Cat}^{\mathrm{op}}$ are Kleisli categories. Similarly, by reversing the 2-cells, one obtains the usual theory of comonads.

One can also define *adjunctions* in a 2-category $\mathscr{K}$, and just as in the classical case $\mathscr{K} = \mathsf{Cat}$, every adjunction induces a monad. On the other hand, if the Eilenberg–Moore object A^t for a monad (A,t) exists, then there is a canonical "forgetful arrow" $u^t:A^t \to A$ which has a left adjoint, and this adjunction induces the original monad t. Similarly, if the Kleisli object exists, then the resulting adjunction also induces the original monad t.

The definition of Eilenberg–Moore objects in terms of a representability conditions prompts one to consider whether Eilenberg–Moore objects might be (weighted) limits, and this is in fact the case. The (algebraicists') simplicial category, consisting of the finite ordinals and order-preserving maps, has a strict monoidal structure given by ordinal sum, and so there is a 2-category mnd with a single object $*$ and with hom-category $\mathsf{mnd}(*,*)$ equal to the simplicial category. A 2-functor from mnd to $\mathscr{K}$ is precisely a monad in $\mathscr{K}$, and the second author constructed in [22] a 2-functor $J:\mathsf{mnd} \to \mathsf{Cat}$ for which the Eilenberg–Moore object of a monad in $\mathscr{K}$ was just the J-weighted limit of the corresponding 2-functor from mnd to $\mathscr{K}$. (This idea is essentially contained in [16], but the language of weighted limits was not available at that time to express it.) It was further proved in [22] that the "weight" J is finite, in the sense that J-limits exist provided that $\mathscr{K}$ has finite products, equalizers, and cotensors with the arrow category 2; this is equivalent to its being finite in the sense of [9].

Since Eilenberg–Moore objects are weighted limits, one can consider the *free completion under Eilenberg–Moore objects* of a 2-category $\mathscr{K}$. This is defined to be a 2-category $\mathsf{EM}(\mathscr{K})$ with Eilenberg–Moore objects, and a 2-functor $Z:\mathscr{K} \to \mathsf{EM}(\mathscr{K})$ with the property that for any 2-category $\mathscr{L}$ with Eilenberg–Moore objects, composition with Z induces an equivalence of categories between the 2-functor category $[\mathscr{K},\mathscr{L}]$ and the full subcategory of the 2-functor category $[\mathsf{EM}(\mathscr{K}),\mathscr{L}]$ consisting of those 2-functors which preserve Eilenberg–Moore objects. By the general theory of such completions [8, Chapter 5], we know that Z will be fully faithful, and that to give a right adjoint to $Z : \mathscr{K} \to \mathsf{EM}(\mathscr{K})$ is precisely to give a choice, for each monad in $\mathscr{K}$, of an Eilenberg–Moore object of the monad. In the light of Proposition 0.1 one is led to ask whether $\mathsf{Mnd}(\mathscr{K})$ might not itself be the free completion $\mathsf{EM}(\mathscr{K})$. This is not the case; indeed $\mathsf{Mnd}(\mathscr{K})$ need not even possess Eilenberg–Moore objects. Our first main contribution is an explicit description of $\mathsf{EM}(\mathscr{K})$. It will turn out to have the same objects and 1-cells as $\mathsf{Mnd}(\mathscr{K})$, but different 2-cells; there will, however, be a 2-functor $E : \mathsf{Mnd}(\mathscr{K}) \to \mathsf{EM}(\mathscr{K})$ which is the identity on objects and 1-cells.

The second major aspect of the formal theory of monads developed in [21] concerns distributive laws [1]. Given monads (A,t) and (A,s) on an object A, a *distributive law* of t over s consists of a 2-cell $\lambda : ts \to st$ satisfying four equations which may be found in [1], but which amount to the fact that (s,λ) is a monad morphism from (A,t) to (A,t)—that is, a 1-cell in $\mathsf{Mnd}(\mathsf{Cat})$—and that η and μ are monad transformations $1 \to (s,\lambda)$ and $(s,\lambda)^2 \to (s,\lambda)$—that is, 2-cells in $\mathsf{Mnd}(\mathsf{Cat})$. In other words, a distributive law is simply a monad in $\mathsf{Mnd}(\mathsf{Cat})$, which in turn is just an object of $\mathsf{Mnd}(\mathsf{Mnd}(\mathsf{Cat}))$. The key fact about a distributive law $\lambda : ts \to st$ is that it allows one to define a monad structure on the endofunctor st.

This is further analyzed in [21] as follows. Let $2\text{-}\mathsf{Cat}_0$ be the category of 2-categories and 2-functors. There is an endofunctor Mnd of $2\text{-}\mathsf{Cat}$ which sends a 2-category $\mathscr{K}$ to the 2-category $\mathsf{Mnd}(\mathscr{K})$, and is defined on 1- and 2-cells in the evident way. There is a natural transformation Id from the identity endofunctor of $2\text{-}\mathsf{Cat}_0$ to Mnd whose component at $\mathscr{K}$ is the 2-functor $\mathsf{Id} : \mathscr{K} \to \mathsf{Mnd}(\mathscr{K})$. There is also a natural transformation $\mathsf{Comp} : \mathsf{Mnd}\mathsf{Mnd} \to \mathsf{Mnd}$ of which the component at the 2-category Cat is a 2-functor $\mathsf{Mnd}(\mathsf{Mnd}(\mathsf{Cat})) \to \mathsf{Mnd}(\mathsf{Cat})$ whose effect on objects is to send a distributive law to the induced composite monad. Then Id and Comp are the unit and multiplication of a monad Mnd on $2\text{-}\mathsf{Cat}_0$.
By the general properties of completions under limits, there is a 2-functor $\mathsf{Comp} : \mathsf{EM}(\mathsf{EM}(\mathscr{K})) \to \mathsf{EM}(\mathscr{K})$ which is right adjoint to $\mathsf{Id} : \mathsf{EM}(\mathscr{K}) \to \mathsf{EM}(\mathsf{EM}(\mathscr{K}))$. An object of $\mathsf{EM}(\mathsf{EM}(\mathscr{K}))$ we call a *wreath*, and we think of as being a kind of generalized distributive law. (In earlier versions [15,14] of this work, we used the name *extended distributive law* for what we now call wreath.) Our second main contribution is to define wreaths and the composite monad ("wreath product") induced by a wreath, to study their basic properties, and to provide examples illustrating the utility of the notion.

In Section 1 we compute explicitly the free completion under Eilenberg–Moore objects of a 2-category $\mathscr{K}$. In fact, since free completions under colimits are more familiar than completions under the corresponding limits, being formed as full subcategories of presheaf categories, we work with the colimit notion corresponding to Eilenberg–Moore objects, namely Kleisli objects. The free completion under Kleisli objects of a 2-category $\mathscr{K}$ may be formed as the closure $\mathsf{KL}(\mathscr{K})$ of the representables in $[\mathscr{K}^{\mathrm{op}}, \mathsf{Cat}]$ under Kleisli objects; then we may take $\mathsf{EM}(\mathscr{K})$ to be $\mathsf{KL}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$.

In Section 2 we describe $\mathsf{EM}(\mathscr{K})$ for certain particular 2-categories $\mathscr{K}$. We treat in detail the cases $\mathscr{K} = \mathsf{Cat}$ and Span, and, more briefly, the case of an arbitrary 2-category $\mathscr{K}$ with Eilenberg–Moore objects. Finally, we show how the 2-category of generalized multicategories as defined by Hermida [6], and the 2-category of categories enriched in a bicategory $\mathscr{W}$, arise in this context.

In Section 3 we study wreaths and the composite monads they induce. We treat the case $\mathscr{K} = \mathsf{Cat}$ in detail, and then give various examples showing that wreaths actually arise in practice. These examples are diverse, involving group cohomology, Hopf algebras, Kleisli categories, and factorization systems on categories.

There is also an appendix containing an account of “two-dimensional partial maps”, which arise in connection with our construction of $\mathsf{EM}(\mathsf{Span})$. In particular, we prove an old unpublished result of Lawvere [17].

As a general reference on matters 2-categorical, one might consult [11].

1. Free completions under Eilenberg–Moore objects

In this section, we describe the free completion under Eilenberg–Moore objects of a 2-category $\mathscr{K}$; as anticipated in the Introduction, this 2-category $\mathsf{EM}(\mathscr{K})$ will have the same objects and 1-cells as $\mathsf{Mnd}(\mathscr{K})$, but different 2-cells. As proposed in the Introduction, we shall first calculate $\mathsf{KL}(\mathscr{K})$, the free completion under Kleisli objects; then $\mathsf{EM}(\mathscr{K})$ will be given by $\mathsf{KL}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$.

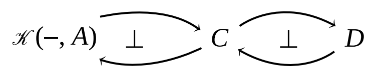
Given a monad (A,t) in $\mathscr{K}$, we write A_t for the Kleisli object; this is defined to be the Eilenberg–Moore object of (A,t), seen as a monad in $\mathscr{K}^{\mathrm{op}}$, and determined by a natural isomorphism

$$\mathscr{K}(A_t,X) \cong \mathscr{K}(A,X)^{\mathscr{K}(t,X)}$$

of categories, where $\mathscr{K}(t,X)$ is the monad on $\mathscr{K}(A,X)$ induced by t.

By [8, Theorem 5.35] the free completion $\Phi(\mathscr{K})$ of a 2-category $\mathscr{K}$ under a class Φ of colimits may be formed as the closure of the representables in $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ under Φ-colimits; these colimits in the functor category $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ are of course formed pointwise. This closure may be constructed by a transfinite process: one starts with the representables, then throws in those objects which are Φ-colimits of representables, then those objects which are Φ-colimits of those, and so on.

In the case at hand, where Φ consists only of (the weight for) Kleisli objects, something very special happens. For if C is the Kleisli object in $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ for a monad on the representable $\mathscr{K}(-,A)$, and D is the Kleisli object for a monad on C, then we have a pair of adjunctions



of Kleisli type; that is, the left adjoints are both (pointwise) bijective on objects. But then in the composite adjunction the left adjoint is once again bijective on objects, and so D is itself the Kleisli object for the monad on $\mathscr{K}(-,A)$ induced by the composite adjunction. In other words, the possibly transfinite process described in the previous paragraph actually terminates after one step. Thus, we may take $\mathsf{KL}(\mathscr{K})$ to be the full subcategory of $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ consisting of those presheaves which are Kleisli objects of monads on representables; here it is important to note that a representable object

$\mathscr{K}(-,A)$ is itself the Kleisli object for a monad on a representable, for instance, the identity monad on $\mathscr{K}(-,A)$.

Finally, one further simplification is possible. Since the universal property of free completions determines them only up to equivalence, we may as well take the objects of $\mathsf{KL}(\mathscr{K})$ to be the monads themselves. Then we take $\mathsf{KL}(\mathscr{K})$ to be the 2-category whose objects are the monads in $\mathscr{K}$, and which has a fully faithful 2-functor $W:\mathsf{KL}(\mathscr{K}) \to [\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ sending the object (A,t) of $\mathsf{KL}(\mathscr{K})$ to the Kleisli object of the monad $\mathscr{K}(-,t)$ on $\mathscr{K}(-,A)$ induced by the Yoneda embedding $Y:\mathscr{K} \to [\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$.

We shall now derive an explicit description of $\mathsf{KL}(\mathscr{K})$. A 1-cell in $\mathsf{KL}(\mathscr{K})$ from (A,t) to (B,s) is a 1-cell in $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ from $\mathscr{K}(-,A)_{\mathscr{K}(-,t)}$ to $\mathscr{K}(-,B)_{\mathscr{K}(-,s)}$; which, by the universal property of $\mathscr{K}(-,A)_{\mathscr{K}(-,t)}$ is the same thing as an algebra for the monad $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}](\mathscr{K}(-,t),\mathscr{K}(-,B)_{\mathscr{K}(-,s)})$ on $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}](\mathscr{K}(-,A),\mathscr{K}(-,B)_{\mathscr{K}(-,s)})$. Now by the Yoneda lemma this last category is isomorphic to $\mathscr{K}(A,B)_{\mathscr{K}(A,s)}$, which has 1-cells from A to B as objects, and 2-cells $f \to sg$ as arrows from f to g, and the usual Kleisli composition. The monad on $\mathscr{K}(A,B)_{\mathscr{K}(A,s)}$ corresponding to $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}](\mathscr{K}(-,t),\mathscr{K}(-,B)_{\mathscr{K}(-,s)})$ under this isomorphism we call $\mathscr{K}(t,B)_{\mathscr{K}(t,s)}$; its endofunctor part sends an object $f:A \to B$ of $\mathscr{K}(A,B)_{\mathscr{K}(A,s)}$ to ft, and a morphism $\rho: f \to sg$ to $\rho t: ft \to sgt$; the component at f of the multiplication is $\eta ft.f\mu: ftt \to sft$, while the component at f of the unit is $\eta ft.f\eta: f \to sgt$.

Thus, a 1-cell in $\mathsf{KL}(\mathscr{K})$ from (A,t) to (B,s) consists of a 1-cell $f:A \to B$ in $\mathscr{K}$ equipped with a 2-cell $\phi: ft \to sf$, satisfying the associative and unit laws

$$
\begin{array}{ccccc}
ftt & \xrightarrow{\phi t} & sft & \xrightarrow{s\phi} & ssf \\
{\scriptstyle f\mu}\downarrow & & & & \downarrow{\scriptstyle \mu f} \\
ft & & \xrightarrow[\phi]{\qquad\qquad} & & sf
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & f & \\
{\scriptstyle f\eta}\swarrow & & \searrow{\scriptstyle \eta f} \\
ft & \xrightarrow[\phi]{} & sf
\end{array}
$$

for a $\mathscr{K}(t,B)_{\mathscr{K}(t,s)}$-algebra.

Given 1-cells $(f,\phi),(g,\psi):(A,t) \to (B,s)$ in $\mathsf{KL}(\mathscr{K})$, a 2-cell from (f,ϕ) to (g,ψ) should now be a 2-cell in $[\mathscr{K}^{\mathrm{op}},\mathsf{Cat}]$ between the 1-cells $W(f,\phi)$ and $W(g,\psi)$. This amounts to a morphism from (f,ϕ) to (g,ψ), seen as algebras for the monad $\mathscr{K}(t,B)_{\mathscr{K}(t,s)}$. To give a morphism from f to g in $\mathscr{K}(A,B)_{\mathscr{K}(A,s)}$ is to give a 2-cell $\rho: f \to sg$ in $\mathscr{K}$; to ask that it be an algebra morphism is to ask for commutativity of

$$
\begin{array}{ccccc}
ft & \xrightarrow{\phi} & sf & \xrightarrow{s\rho} & ssg \\
{\scriptstyle \rho t}\downarrow & & & & \downarrow{\scriptstyle \mu g} \\
sgt & \xrightarrow[s\psi]{} & ssg & \xrightarrow[\mu g]{} & sg.
\end{array}
$$

The vertical composite of the 2-cells $\rho:(f,\phi) \to (g,\psi)$ and $\tau:(g,\psi) \to (h,\theta)$ is the 2-cell

$$f \xrightarrow{\rho} sg \xrightarrow{s\tau} ssh \xrightarrow{\mu h} sh$$

while the horizontal composite of $\rho:(f,\phi)\to(g,\psi)$ and $\rho':(f',\phi')\to(g,\psi'):(B,s)\to(C,r)$ is the 2-cell

$$f'f \xrightarrow{f'\rho} f'sg \xrightarrow{\phi' g} rf'g \xrightarrow{rrg} rrg'g \xrightarrow{\mu g' g} rg'g$$

This completes the explicit description of $\mathsf{KL}(\mathscr{K})$.

We may now find $\mathsf{EM}(\mathscr{K})$ as $\mathsf{KL}(\mathscr{K}^{\mathrm{op}})^{\mathrm{op}}$; an object is of course still a monad in $\mathscr{K}$; a 1-cell from (A,t) to (B,s) is now a 1-cell $f:A\to B$ equipped with a 2-cell $\phi:sf\to ft$ satisfying the two conditions

$$\begin{array}{ccccc} ssf & \xrightarrow{s\phi} & sft & \xrightarrow{\phi t} & ftt \\ {\scriptstyle \mu f}\downarrow & & & & \downarrow{\scriptstyle f\mu} \\ sf & & \xrightarrow[\phi]{} & & ft \end{array} \qquad \begin{array}{ccc} & f & \\ {\scriptstyle \eta f}\swarrow & & \searrow{\scriptstyle f\eta} \\ sf & \xrightarrow[\phi]{} & ft \end{array}$$

while a 2-cell from (f,ϕ) to (g,ψ) is a 2-cell $\rho:f\to gt$ in $\mathscr{K}$ satisfying

$$\begin{array}{ccccc} sf & \xrightarrow{\phi} & ft & \xrightarrow{\rho t} & gtt \\ {\scriptstyle s\rho}\downarrow & & & & \downarrow{\scriptstyle g\mu} \\ sgt & \xrightarrow[\psi t]{} & gtt & \xrightarrow[g\mu]{} & gt \end{array}$$

We have just described what we call the *reduced form* of the 2-cells in $\mathsf{EM}(\mathscr{K})$; there is also an *unreduced form*, according to which a 2-cell from (f,ϕ) to (g,ψ) consists of a 2-cell $\hat{\rho}:sf\to gt$ now satisfying two conditions:

$$\begin{array}{ccccc} ssf & \xrightarrow{s\hat{\rho}} & sgt & \xrightarrow{\psi t} & gtt \\ {\scriptstyle \mu f}\downarrow & & & & \downarrow{\scriptstyle g\mu} \\ sf & & \xrightarrow[\hat{\rho}]{} & & gt \end{array} \qquad \begin{array}{ccccc} sft & \xrightarrow{s\phi} & ssf & \xrightarrow{\mu f} & sf \\ {\scriptstyle \hat{\rho} t}\downarrow & & & & \downarrow{\scriptstyle \hat{\rho}} \\ gtt & & \xrightarrow[g\mu]{} & & gt \end{array}$$

The bijection between reduced and unreduced forms sends $\hat{\rho}$ to $\hat{\rho}.\eta f$, and ρ to $g\mu.\psi t.s\rho$. This situation is precisely analogous to the two descriptions of the Kleisli category for a monad t on a category A; as having the same objects as A and arrows from a to b given by arrows $a\to tb$ in A, or as the full subcategory of the Eilenberg–Moore category consisting of the free algebras. In our terminology the former is the reduced form, and the latter is the unreduced form. (Of course, the 2-cells in $\mathsf{KL}(\mathscr{K})$ also have an unreduced form, along with the reduced form already described.)

We can now see, as promised, that $\mathsf{EM}(\mathscr{K})$ has the same objects and 1-cells as $\mathsf{Mnd}(\mathscr{K})$, and indeed there is a 2-functor $E:\mathsf{Mnd}(\mathscr{K})\to\mathsf{EM}(\mathscr{K})$ which is the identity on objects and 1-cells, and sends a 2-cell $\rho:(f,\phi)\to(g,\psi)$ to $g\eta.\rho:f\to gt$.

The (restricted) Yoneda embedding is the map $\mathsf{Id}:\mathscr{K}\to\mathsf{EM}(\mathscr{K})$ sending an object A to the identity monad on A; if $\mathscr{L}$ is a 2-category with Eilenberg–Moore objects then for every 2-functor $F:\mathscr{K}\to\mathscr{L}$ there is an essentially unique Eilenberg–Moore-object-preserving 2-functor from $\mathsf{EM}(\mathscr{K})$ to $\mathscr{L}$ whose composite with Id is isomorphic to F.

It is now an immediate consequence, as observed in the Introduction, that $\mathscr{K}$ has Eilenberg–Moore objects if and only if $\mathsf{Id}:\mathscr{K}\to\mathsf{EM}(\mathscr{K})$ has a right adjoint, and that there is a bijection between choices of a right adjoint to $\mathsf{Id}:\mathscr{K}\to\mathsf{EM}(\mathscr{K})$ and choices of Eilenberg–Moore objects for each monad in $\mathscr{K}$.

Remark 1.1. We have defined $\mathsf{EM}(\mathscr{K})$ and $\mathsf{KL}(\mathscr{K})$ only when $\mathscr{K}$ is a 2-category, but it is straightforward to extend this definition to include the case of bicategories—one simply mimics the explicit description of $\mathsf{EM}(\mathscr{K})$ or $\mathsf{KL}(\mathscr{K})$—and it is now clear that if a bicategory $\mathscr{B}$ is biequivalent to a 2-category $\mathscr{K}$, then also $\mathsf{EM}(\mathscr{B})$ is biequivalent to $\mathsf{EM}(\mathscr{K})$, and $\mathsf{KL}(\mathscr{B})$ to $\mathsf{KL}(\mathscr{K})$.

Although it would be possible to develop the whole formal theory of monads in the context of general bicategories, we have chosen to work in the context of 2-categories, so as to have available to us the theory of free (co)limit completions in the enriched context of [8].

2. Examples of $\mathsf{EM}(\mathscr{K})$ for various 2-categories $\mathscr{K}$

In this section we give an interpretation of $\mathsf{EM}(\mathscr{K})$ in two cases: where $\mathscr{K}$ is the 2-category Cat, and where $\mathscr{K}$ is the 2-category Span. We also use the first case to interpret $\mathsf{EM}(\mathscr{K})$ where $\mathscr{K}$ is an arbitrary 2-category with Eilenberg–Moore objects.

2.1. $\mathsf{EM}(\mathsf{Cat})$

First we consider the case where $\mathscr{K}$ is the 2-category Cat. Then an object of $\mathsf{EM}(\mathsf{Cat})$ is of course just a monad in the usual sense. Given monads (A,t) and (B,s), a morphism $(f,\phi):(A,t)\to(B,s)$ determines a functor $\bar{f}:A^t\to B^s$, sending a t-algebra $(a,\rho:ta\to a)$ to the s-algebra $(fa, f\rho.\phi a:sfa\to fa)$. In fact, this process is part of a bijection, enabling us to identify 1-cells from (A,t) to (B,s) in $\mathsf{EM}(\mathsf{Cat})$, or equally in $\mathsf{Mnd}(\mathsf{Cat})$, with commutative squares

$$\begin{array}{ccc} A^t & \xrightarrow{\ \bar{f}\ } & B^s \\ {\scriptstyle u^t}\downarrow & & \downarrow{\scriptstyle u^s} \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

in Cat; recall that u^t and u^s denote the forgetful functors from the relevant Eilenberg–Moore categories. Similarly, 2-cells in $\mathsf{Mnd}(\mathsf{Cat})$ may be identified with pairs of natural transformations as in

$$\begin{array}{ccc} A^t & \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\bar{\rho}}} & B^s \\ {\scriptstyle u^t}\downarrow & & \downarrow{\scriptstyle u^s} \\ A & \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow\rho}} & B; \end{array}$$

this diagram is to be understood as expressing the equality $u^s\bar{\rho}=\rho u^t$. Thus, we may view $\mathsf{Mnd}(\mathsf{Cat})$ as a full sub-2-category of the arrow 2-category $\mathsf{Cat}^{\mathbf{2}}$.

A 2-cell $(f,\bar{f}) \to (g,\bar{g})$ in $\mathsf{EM}(\mathsf{Cat})$, however, turns out to be an arbitrary natural transformation $\bar{f} \to \bar{g}$. Given a 2-cell $\rho : f \to gt$ in reduced form, the component at $(a, \alpha : ta \to a)$ of the corresponding natural transformation $\bar{f} \to \bar{g}$ is the homomorphism of s-algebras $g\alpha.\rho a : fa \to ga$.

Thus the objects of $\mathsf{EM}(\mathsf{Cat})$ are the monads in Cat, the 1-cells from (A,t) to (B,s) are the pairs $(f,\bar{f})$ of 1-cells in Cat satisfying $u^s\bar{f} = fu^t$, and the 2-cells from $(f,\bar{f})$ to $(g,\bar{g})$ are the 2-cells $\bar{f} \to \bar{g}$.

The 2-functor $E : \mathsf{Mnd}(\mathsf{Cat}) \to \mathsf{EM}(\mathsf{Cat})$ is now the identity on objects and 1-cells, and acts on the 2-cell $(\rho,\bar{\rho})$ by forgetting ρ.

2.2. $\mathsf{EM}(\mathscr{K})$ *for a 2-category* $\mathscr{K}$ *with Eilenberg–Moore objects*

The analysis of the previous section carries over unchanged to the case of a 2-category $\mathscr{K}$ with Eilenberg–Moore objects. For such a $\mathscr{K}$, we continue to write A^t for the Eilenberg–Moore object of a monad (A,t), and to write $u^t : A^t \to A$ for the canonical "forgetful" 1-cell. Then the objects of $\mathsf{EM}(\mathscr{K})$ are once again the monads in $\mathscr{K}$, the 1-cells from (A,t) to (B,s) are once again the pairs $(f,\bar{f})$ of 1-cells satisfying $u^s\bar{f} = fu^t$, and the 2-cells from $(f,\bar{f})$ to $(g,\bar{g})$ are the 2-cells $(\bar{f} \to \bar{g})$.

Once again $\mathsf{Mnd}(\mathscr{K})$ may be viewed as a full sub-2-category of the 2-category $\mathscr{K}^{\mathbf{2}}$, consisting of those objects of $\mathscr{K}^{\mathbf{2}}$ of the form $u^t : A^t \to A$ for some monad (A,t). In fact, this was the motivation for the definition in [13] of the Gray-category of pseudomonads in a Gray-category.

2.3. $\mathsf{EM}(\mathsf{Span})$

We now take $\mathscr{K}$ to be the bicategory Span of sets and spans. Here we have two problems to face. The first is that Span is not actually a 2-category, but only a bicategory, since composition, which is defined using pullbacks, is not strictly associative, but only associative up to isomorphism. The second problem is that the hom-categories $\mathsf{Span}(X,Y)$ are not small: indeed $\mathsf{Span}(X,Y) \cong \mathsf{Set}/(X \times Y) \simeq \mathsf{Set}^{X\times Y}$.

Thus far, we have implicitly been working with a category Set of small sets, and the complete and cocomplete symmetric monoidal closed category Cat of all categories in Set; then 2-category has meant category enriched in Cat. We now need to replace Cat by a complete and cocomplete symmetric monoidal closed category CAT which contains the categories $\mathsf{Set}/(X\times Y)$ for all X and Y in Set; since CAT is to be complete, it will contain $\mathsf{Set}/(X\times Y)$ provided that it contains Set. We shall use the epithet *small* to emphasize that a category A lies in Cat. All the previous theory will carry over if we interpret 2-category to mean category enriched in CAT. This deals with the second problem.

As for the first problem, we have already mentioned in Remark 1.1 that one can define $\mathsf{EM}(\mathscr{K})$ and $\mathsf{KL}(\mathscr{K})$ when $\mathscr{K}$ is only a bicategory, but in fact it turns out to be convenient to work with a 2-category $\mathscr{K}$ which is biequivalent to Span. Our definition of $\mathscr{K}$ uses the "Fam" construction; in an appendix to this paper we recall the details of the construction, along with a proof that the 2-category that we shall now describe is indeed biequivalent to Span. We write COPR for the 2-category of categories with coproducts, coproduct-preserving functors, and natural transformations, we write kl(Fam) for the full sub-2-category of COPR consisting of the categories with coproducts of the form Fam(A) for a small category A, and we write $\mathsf{kl(Fam)_D}$ for the full sub-2-category of COPR consisting of the categories with coproducts of the form Fam(A) for a small discrete category A. It is this last 2-category $\mathsf{kl(Fam)_D}$ which is biequivalent to Span. The biequivalence maps a set X, seen as an object of Span, to Fam(X), where the set X is here regarded as a discrete category. (We shall always regard the small category A as being given along with the category Fam(A) when dealing with objects of kl(Fam) or $\mathsf{kl(Fam)_D}$; strictly speaking, we should simply take the small categories—or, in the case of $\mathsf{kl(Fam)_D}$, the small discrete categories—as objects, but we shall ignore such subtleties.)

We shall show that kl(Fam) is the free completion $\mathsf{KL(kl(Fam)_D)}$ of $\mathsf{kl(Fam)_D}$ under Kleisli objects. Then EM(Span) will be $\mathsf{KL(Span^{op})^{op}}$, which is isomorphic to $\mathsf{KL(Span)^{op}}$, since $\mathsf{Span^{op}} \cong \mathsf{Span}$. The biequivalence $\mathsf{Span} \sim \mathsf{kl(Fam)_D}$ will then allow us to deduce a biequivalence between EM(Span) and $\mathsf{kl(Fam)^{op}}$.

To prove that kl(Fam) is the free completion of $\mathsf{kl(Fam)_D}$ under Kleisli objects, it will suffice, by [8, Proposition 5.62], to show that kl(Fam) has Kleisli objects, that every object of kl(Fam) is the Kleisli object of a monad in $\mathsf{kl(Fam)_D}$, and that the representable functors $\mathsf{kl(Fam)(Fam}(X),-) : \mathsf{kl(Fam)} \to \mathsf{CAT}$ preserve Kleisli objects for all objects Fam(X) in $\mathsf{kl(Fam)_D}$.

If the category A has coproducts, and the functor $f : A \to B$ is a left adjoint which is bijective on objects, then B has coproducts, f preserves coproducts, and for any category C with coproducts a functor $g : B \to C$ preserves coproducts if and only if gf does so; one easily deduces that COPR has Kleisli objects, and that the forgetful 2-functor from COPR to CAT preserves them. In fact, the full sub-2-category $\mathsf{kl(Fam)_D}$ of COPR is closed under the Kleisli construction: the Kleisli object of a monad t on Fam(A) is Fam(B) where B is the category with the same objects as A and with hom-sets given by $B(a,a') = \mathsf{Fam}(A)(a,ta')$. This proves the existence of Kleisli objects in kl(Fam), and that the inclusion into CAT preserves them.

Next we show that every object of kl(Fam) is the Kleisli object of some monad in $\mathsf{kl(Fam)_D}$. Given an object Fam(A) of kl(Fam), we write A_0 for the set of objects of A, seen as a discrete category, and $e : A_0 \to A$ for the inclusion functor. Then the functor $\mathsf{Fam}(e) : \mathsf{Fam}(A_0) \to \mathsf{Fam}(A)$ has a right adjoint, which takes the family $(a_i)_{i \in I}$ to the family $(\mathrm{dom}(\alpha))_{\mathrm{cod}(\alpha)=a_i, i \in I}$. Furthermore, this right adjoint clearly preserves coproducts, so that the entire adjunction lives inside the 2-category kl(Fam). Finally, Fam(e) is bijective on objects, so that Fam(A) is the Kleisli object of the induced monad in CAT, and so also in kl(Fam). Thus, Fam(A) is the Kleisli object in kl(Fam) of the monad on Fam(A_0) in $\mathsf{kl(Fam)_D}$ induced by the above adjunction.

We have seen that kl(Fam) has Kleisli objects, and that every object is the Kleisli object of a monad in kl(Fam); it remains to show that $\mathsf{kl(Fam)(Fam}(X),-) : \mathsf{kl(Fam)} \to \mathsf{CAT}$ preserves Kleisli objects for all Fam(X) in $\mathsf{kl(Fam)_D}$. To see this we observe that $\mathsf{kl(Fam)(Fam}(X),-)$ is isomorphic to the composite of the inclusion $\mathsf{kl(Fam)} \to \mathsf{CAT}$ and the endo-2-functor $(\ )^X$ of CAT sending a category A to the product A^X of X copies of A. We have already seen that the inclusion $\mathsf{kl(Fam)} \to \mathsf{CAT}$

preserves Kleisli objects; while $(\,)^X$ preserves adjunctions (like any 2-functor) and functors which are bijective on objects, so preserves Kleisli objects. It now follows that $\mathsf{kl}(\mathsf{Fam})(\mathsf{Fam}(X), -)$ preserves Kleisli objects for all objects $\mathsf{Fam}(X)$ of $\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}}$.

This completes the proof that $\mathsf{kl}(\mathsf{Fam})$ is the free completion under Kleisli objects of $\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}}$; giving:

Proposition 2.1. *The free completion under Kleisli objects of* Span *is* (*biequivalent to*) *the full sub-2-category* $\mathsf{kl}(\mathsf{Fam})$ *of* COPR, *consisting of those categories with coproducts of the form* $\mathsf{Fam}(A)$ *for a small category* A.

Before leaving this topic, we make a few remarks linking the approach to computing the free completion $\mathsf{KL}(\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}})$ used in this section, based on the abstract characterization of [8, Proposition 5.62], with the concrete description of $\mathsf{KL}(\mathscr{K})$ for an arbitrary $\mathscr{K}$, given in the Section 1, and going back to the description of free completions under classes of colimits as full subcategories of presheaf categories.

According to the description given in Section 1, an object of $\mathsf{KL}(\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}})$ should be a monad in $\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}}$; that is, a monad in Span. According to the description of this section, an object of $\mathsf{KL}(\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}})$ should be a category of the form $\mathsf{Fam}(A)$ for a small category A, but we may as well identify this with the small category A itself. From these two approaches we see in a new light the old observation [2] of Bénabou that a monad in Span is precisely a category. Recall that a monad in Span consists of a set C_0 (the set of objects), a span

$$\begin{array}{ccc} & C_1 & \\ {\scriptstyle d}\swarrow & & \searrow{\scriptstyle c} \\ C_0 & & C_0 \end{array}$$

in which we take C_1 to be the set of arrows, and d and c to be the functions picking out the domain and codomain of an arrow. Then the unit of the monad is a morphism of spans from the identity span on C_0 to $(d,c)\colon C_0 \nrightarrow C_0$; that is a function $i\colon C_0 \to C_1$ satisfying $di = ci = 1_{C_0}$. This we interpret as the function sending an object of the category to the identity arrow on that object. Finally, the multiplication of the monad is a morphism of spans from $(d,c)(d,c)$ to (d,c); that is, a function $m\colon C_2 \to C_1$, where C_2 is the object appearing in the pullback

$$\begin{array}{ccc} C_2 & \xrightarrow{p} & C_1 \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle d} \\ C_1 & \xrightarrow{c} & C_0 \end{array}$$

satisfying $dm = dq$ and $cm = cp$, subject to the usual associative and unit laws for a monad, corresponding to the associative and identity laws for a category.

If C and D are categories, viewed as monads in Span, then a morphism of monads from C to D consists of a span $(u,v): C_0 \nrightarrow D_0$ equipped with a morphism of spans $(u,v)(d,c) \to (d,c)(u,v)$, satisfying two conditions. In the special case where u is the identity, this amounts to a function $f_0 : C_0 \to D_0$, and a function $f_1 : C_1 \to D_1$ with $df_1 = f_0 d$ and $cf_1 = f_0 c$, and the two conditions now say precisely that f_1 preserves composition and identities, so that f_0 and f_1 are the object-part and arrow-part of a functor $f : C \to D$. Finally, a 2-cell in KL(Span) between such monad morphisms is precisely a natural transformation between the corresponding functors; note here that if one takes 2-cells in Mnd(Span) (that is, monad transformations) rather 2-cells in KL(Span) one does *not* recover the natural transformations, as has been observed by various authors. Thus, Cat is the locally full sub-bicategory of KL(Span) consisting of all the objects, and those morphisms $(f,\phi):(A,t) \to (B,s)$ for which the left leg of the span f is the identity. This now explains how to extend the correspondence between monads in Span and categories, to deal with both functors and natural transformations.

We can also approach this via the 2-category kl(Fam), rather than Span. The biadjoint Fam : CAT → COPR restricts to give a 2-functor Fam : Cat → kl(Fam), and this latter 2-functor exhibits Cat as a sub-2-category of kl(Fam) which has the same objects, and is *locally full*, meaning that the functors Fam : Cat$(A,B) \to$ kl(Fam)(Fam(A), Fam(B)) are fully faithful for all objects A and B of Cat. Thus, up to biequivalence, we can view Cat as living inside KL(Span) as a locally full sub-bicategory, containing all the objects.

2.4. Generalized multicategories

In the last section we saw how to see Cat as living inside KL(Span), by considering only certain arrows. One way to approach this result abstractly is based on the notion of *pro-arrow equipment* [25,26] of Wood. For our purposes, a pro-arrow equipment can be taken to be a homomorphism of bicategories $(\ )_* : \mathscr{C} \to \mathscr{B}$ which is the identity on objects. We then define KL$(\mathscr{B},\mathscr{C})$ to be the following modification of KL$(\mathscr{B})$. Its objects are those of KL$(\mathscr{B})$, and its 1-cells $(A,t) \to (B,s)$ are pairs (f,ϕ) where $f : A \to B$ is a 1-cell in $\mathscr{C}$, and ϕ is a 2-cell in $\mathscr{B}$ for which (f_*,ϕ) is a 1-cell in KL$(\mathscr{B})$. Finally, the 2-cells from (f,ϕ) to (g,ψ) are just the 2-cells $(f_*,\phi) \to (g_*,\psi)$ in KL$(\mathscr{B})$. Now KL$(\mathscr{B},\mathscr{C})$ is precisely what was called *HOM*$(_*)$ in [26, p. 31]. In the last section we saw how to identify Cat with KL(Span, Set).

If $\mathscr{E}$ is any category with pullbacks, and T is a cartesian monad on $\mathscr{E}$ (that is, a monad whose endofunctor part preserves pullbacks, and whose unit and multiplication have the property that each of their naturality squares are pullbacks) then we can, following [6,18], define the bicategory Span$_T(\mathscr{E})$ of *generalized spans* in $\mathscr{E}$. This has the same objects as $\mathscr{E}$, while a 1-cell from A to B is a span from TA to B, and a 2-cell is a morphism of such spans. Hermida and Leinster then define a *generalized multicategory* in $\mathscr{E}$ to be a monad in Span$_T(\mathscr{E})$. If T is the free-monoid monad then one speaks simply of a multicategory in $\mathscr{E}$, while if moreover $\mathscr{E} =$ Set, then these are just the multicategories in the usual sense. Now (Span$_T(\mathscr{E}), \mathscr{E}$)

is a pro-arrow equipment, so we can form $\mathsf{KL}(\mathsf{Span}_T(\mathscr{E}),\mathscr{E})$, and if T is the free-monoid monad then this is just the 2-category of multicategories, as defined in [6].

2.5. Categories enriched in a bicategory

Given a bicategory $\mathscr{W}$ in which the hom-categories are cocomplete, and composition on either side is cocontinuous, a bicategory $\mathscr{W}$-Mat was defined in [3] in which a monad is precisely a category enriched in the bicategory $\mathscr{W}$. If $\mathsf{Ob}\mathscr{W}$ is the set of objects of $\mathscr{W}$, then there is a homomorphism of bicategories $(\ )_* : \mathsf{Set}/\mathsf{Ob}\mathscr{W} \to \mathscr{W}\text{-}\mathsf{Mat}$ which is the identity on objects. It was further shown in [3] that the 2-category $\mathscr{W}$-Cat is what we have called $\mathsf{KL}(\mathscr{W}\text{-}\mathsf{Mat}, \mathsf{Set}/\mathsf{Ob}\mathscr{W})$.

3. Wreaths

From the universal property of the free completion $\mathsf{EM}(\mathscr{K})$, one immediately obtains a pseudomonad EM on the 2-category 2-Cat of 2-categories, 2-functors, and 2-natural transformations, which has the Kock–Zöberlein property [12,27,23] of the "limit-like variance". This will turn out to be a 2-monad; in fact, it is even a 3-monad on the 3-category of 2-categories, 2-functors, 2-natural transformations, and modifications, but we shall not consider the 3-categorical structure. The (component at a 2-category $\mathscr{K}$ of the) unit $\mathsf{Id} : \mathscr{K} \to \mathsf{EM}(\mathscr{K})$ we have already seen to be the restricted Yoneda embedding. We shall write $\mathsf{Comp} : \mathsf{EM}(\mathsf{EM}(\mathscr{K})) \to \mathsf{EM}(\mathscr{K})$ for the (component at $\mathscr{K}$ of the) multiplication, which takes a monad in $\mathsf{EM}(\mathscr{K})$ to its Eilenberg–Moore object.

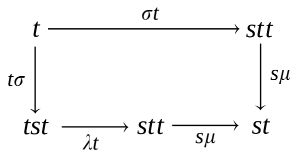
This is very similar to the situation in the final section of [21], where there was a monad Mnd on the category 2-Cat_0 of 2-categories and 2-functors, sending $\mathscr{K}$ to $\mathsf{Mnd}(\mathscr{K})$. An object of $\mathsf{Mnd}(\mathsf{Mnd}(\mathscr{K}))$ was identified with a distributive law in $\mathscr{K}$, and the multiplication $\mathsf{Comp} : \mathsf{Mnd}(\mathsf{Mnd}(\mathscr{K})) \to \mathsf{Mnd}(\mathscr{K})$ was seen to take a distributive law to the induced composite monad. Once again, Mnd is actually a 3-monad, and once again we shall not consider this structure; we will, however, regard it as a 2-monad on 2-Cat, so that it is seen in the same framework as EM.

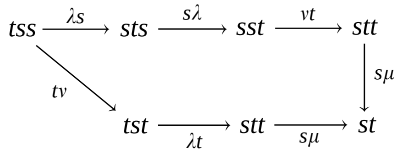
In this section we shall explore the connections between these two situations, by looking at monads in $\mathsf{EM}(\mathscr{K})$ and their Eilenberg–Moore objects.

Given our explicit description of $\mathsf{EM}(\mathscr{K})$, we may simply write down what is a monad in $\mathsf{EM}(\mathscr{K})$. It consists of an object of $\mathsf{EM}(\mathscr{K})$, that is a monad (A,t) in $\mathscr{K}$, an endomorphism of that object, that is a 1-cell $s : A \to A$ in $\mathscr{K}$, and a 2-cell $\lambda : ts \to st$ satisfying the two conditions:

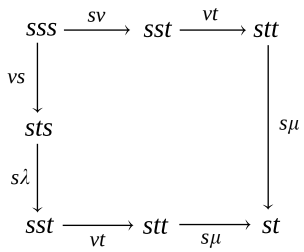
$$
\begin{array}{ccccc}
tts & \xrightarrow{t\lambda} & tst & \xrightarrow{\lambda t} & stt \\
{\scriptstyle \mu s}\downarrow & & & & \downarrow{\scriptstyle s\mu} \\
ts & & \xrightarrow{\quad\lambda\quad} & & st
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & s & & \\
 & {\scriptstyle \eta s}\swarrow & & \searrow{\scriptstyle s\eta} & \\
ts & & \xrightarrow{\quad\lambda\quad} & & st;
\end{array}
$$

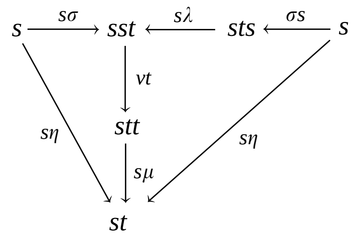
2-cells $1 \to (s,\lambda)$ and $(s,\lambda)(s,\lambda) \to (s,\lambda)$, that is 2-cells $\sigma: 1 \to st$ and $v: ss \to st$ satisfying





subject finally to the associative and unit laws for a monad, which in this case are the conditions





Such a structure is formally similar to a distributive law: s is not a monad, but the 2-cells v and σ behave very much like the multiplication and unit for a monad (which of course they are, if one works in the 2-category $\mathsf{EM}(\mathscr{K})$). We call a monad in $\mathsf{EM}(\mathscr{K})$ a *wreath*. (In earlier versions [15,14] of this paper we used the term *extended distributive law*.) Every distributive law induces a wreath: given monads (A,t) and (A,s) and a distributive law $\lambda: ts \to st$ one takes v and σ to be the maps

$$ss \xrightarrow{\mu} s \xrightarrow{s\eta} st \qquad 1 \xrightarrow{\eta} s \xrightarrow{s\eta} st$$

One may view this process more abstractly as the value on objects of the 2-functor $\mathsf{Mnd}(E): \mathsf{Mnd}(\mathsf{Mnd}(\mathscr{K})) \to \mathsf{Mnd}(\mathsf{EM}(\mathscr{K}))$.

We can also describe explicitly the composite monad (or "wreath product") induced by a wreath: in the above notation it is the monad whose endomorphism part is st, whose unit is σ, and whose multiplication is

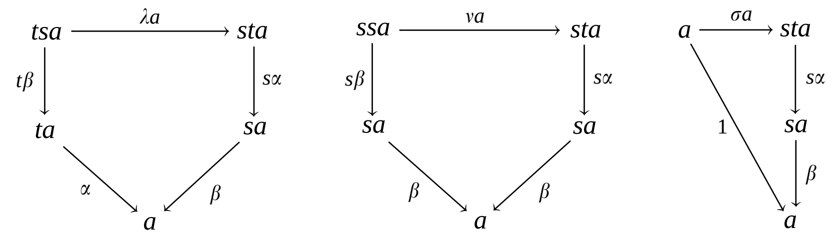
$$stst \xrightarrow{s\lambda t} sstt \xrightarrow{ss\mu} sst \xrightarrow{vt} stt \xrightarrow{s\mu} st$$

With this explicit description one can now verify that this makes EM into a (strict) 2-monad, and that $E: \mathsf{Mnd} \to \mathsf{EM}$ is a strict morphism of 2-monads, in the sense of [10]. The 2-monads with the Kock–Zöberlein property of the limit-like variance were called *colax-idempotent* in [10].

In the classical case of distributive laws in Cat, Beck [1] showed the equivalence of three structures on a pair of monads (A,t) and (A,s): (i) a distributive law $\lambda: ts \to st$, (ii) a lifting of the monad s on A to a monad $\bar{s}$ on A^t, and (iii) a monad structure on the endofunctor st which is compatible in a suitable sense with those on s and t. In the case of a general 2-category $\mathscr{K}$, the equivalence between (i) and (iii) holds unchanged, while the equivalence of these with (ii) holds provided that $\mathscr{K}$ has Eilenberg–Moore objects, so that (ii) makes sense.

When we turn to wreaths, it is clear from the description of EM(Cat) given Section 2.1 that to give a wreath in Cat is equivalent to giving a monad (A,t), endofunctors $s: A \to A$ and $\bar{s}: A^t \to A^t$ satisfying $u^t\bar{s} = su^t$, and a monad structure on the endofunctor $\bar{s}$. Thus not the whole monad on A^t, but only the endofunctor part is lifted from A. Once again, the case of wreaths in an arbitrary 2-category $\mathscr{K}$ with Eilenberg–Moore objects is identical. We shall leave to the reader the formulation of an analog for wreaths of condition (iii).

Another key aspect of Beck's theory of distributive laws is the description of the algebras for the composite monad induced by a distributive law. A similar description is possible in the case of wreaths; for simplicity, we treat only the case $\mathscr{K} = \mathsf{Cat}$. In the above notation, an st-algebra consists of an object a of A, equipped with a morphism $\alpha: ta \to a$ making a into a t-algebra, and a morphism $\beta: sa \to a$ satisfying the following three conditions:



This gives:

Proposition 3.1. *For a wreath in* Cat *between a monad* (A,t) *and an endofunctor* (A,s), *there is an isomorphism* $A^{st} \cong (A^t)^{\bar{s}}$ *where* $\bar{s}$ *is the induced monad on* A^t.

(The reader familiar with the theory of polyads [4] will immediately notice that an st-algebra is precisely an algebra for the polyad consisting of (that is, generated by) the endofunctors s and t, and the natural transformations μ, η, λ, ν, and σ.)

We now give four examples, indicating how wreaths and the composite monads they induce arise in practice.

Example 3.2. This concerns a wreath in CAT on the object Set. Let G be a (multiplicative) group and A an (additive) abelian group, with a homomorphism $G^{\mathrm{op}} \to \mathrm{Aut}(A)$, which we think of as a right action of G on A. The group structure of A defines a monad $A \times -$ on Set which plays the role of the monad t; in the role of the 1-cell s is the endofunctor $G \times -$ of Set. Then $\lambda: ts \to st$ is the natural transformation $A \times G \times - \to G \times A \times -$ induced by the function

$$A \times G \to G \times A$$

$$(a,g) \mapsto (g, a.g)$$

and $\sigma: 1 \to st$ is the natural transformation $1_{\mathsf{Set}} \to G \times A \times -$ induced by the element $(1,0)$ of $G \times A$.

Finally to define $v: ss \to st$, we suppose given a function $\rho: G \times G \to A$; then v is the natural transformation $G \times G \times - \to G \times A \times -$ induced by the function

$$G \times G \to G \times A$$

$$(g,h) \mapsto (gh, \rho(g,h).$$

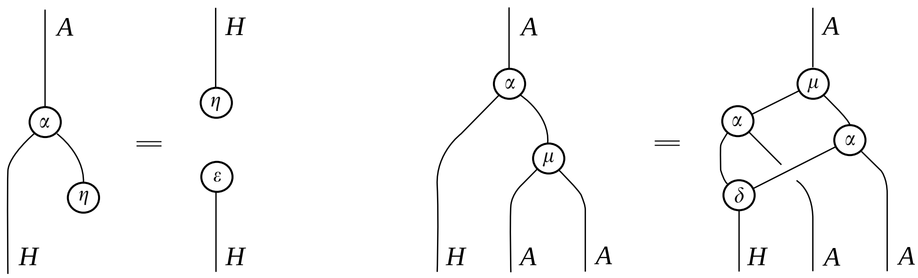
Now these data determine a wreath if and only if the associative and unit laws hold. The unit laws are equivalent to the normalization condition $\rho(g,1) = \rho(1,g) = 0$ on ρ, while associativity is equivalent to the 2-cocycle condition $\rho(gh,k) + \rho(g,h).k = \rho(g,hk) + \rho(h,k)$. Thus, these data determine a wreath if and only if ρ is a normalized 2-cocycle. In this case the composite monad st is $E \times -$, where E is the group appearing in the short exact sequence

$$0 \to A \to E \to G \to 0$$

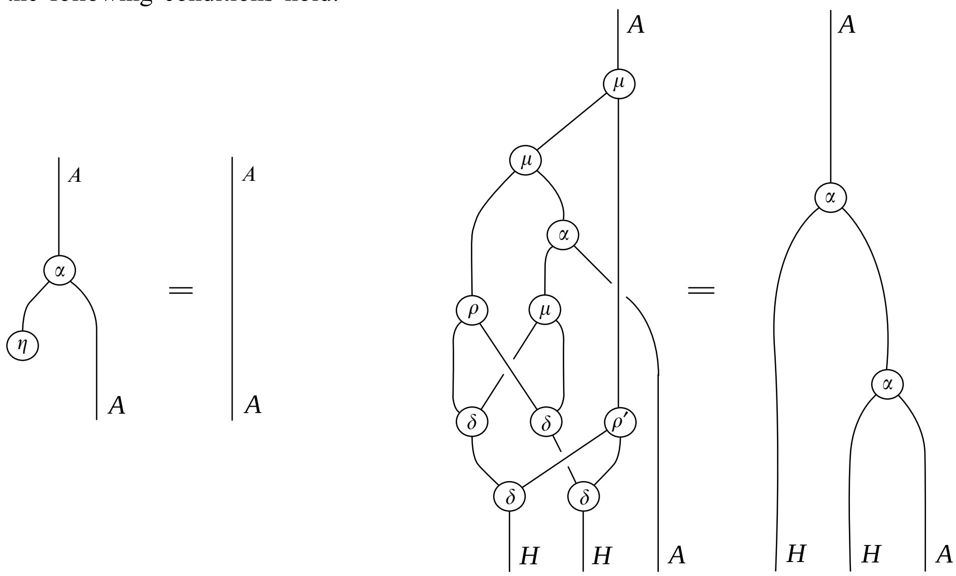
corresponding to the cocycle ρ.

In the previous example, we have chosen to regard the groups A and G as monads on Set (such as $A \times -$), but we could equally well have regarded them as *monoids* in the (cartesian) monoidal category Set, and so as monads in the one-object bicategory Σ(Set) which is its suspension. We leave to the reader the straightforward extension of the definition of wreaths to the case of general bicategories rather than 2-categories, and the verification that the above example can equally be viewed as a wreath in the bicategory Σ(Set). In Example 3.3, essentially a "linearization" of the previous one, we shall work with the monoidal category Vect, leaving to the reader the translation from the language of monoids in Vect to that of monads on Vect.

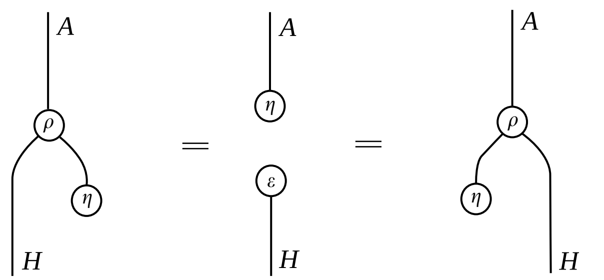
Example 3.3. This example shows how Sweedler's crossed product of Hopf algebras [24] can be described in terms of wreaths. We follow the formulation of [19, Chapter 7], but use the string diagrams of [7]. Let H be a Hopf algebra over k, and let A be a k-algebra; we write μ for the multiplications and η for the units, and write δ and ε for the comultiplication and counit of H. Then H is said to *measure* A, if there is a map $\alpha: H \otimes A \to A$ satisfying the two equations:

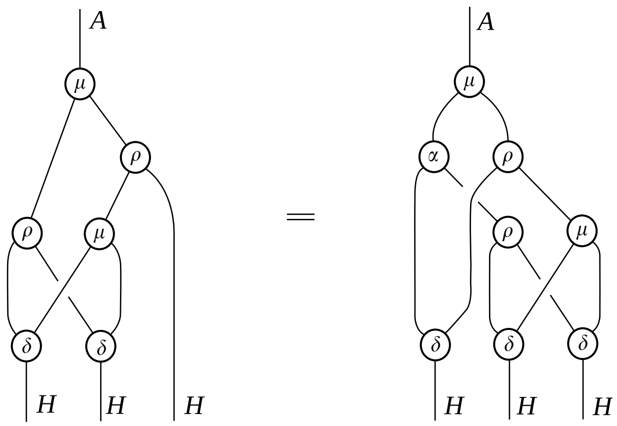


Then A is called a *twisted H-module* if it is further equipped with a map $\rho: H \otimes H \to A$ which is invertible in the convolution algebra $\mathrm{Hom}_k(H \otimes H, A)$, with inverse ρ', and the following conditions hold:

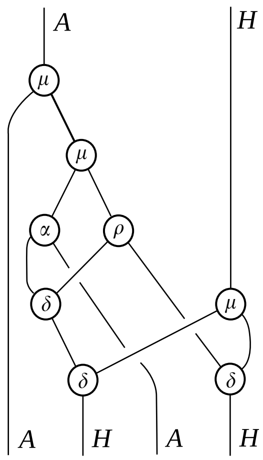


If finally these data satisfy the (normalized) cocycle condition:





then one can define a new algebra, with underlying vector space $A \otimes H$, called the *crossed product* of A and H with respect to the cocycle ρ, and denoted $A\sharp_\rho H$. The unit of this algebra is the tensor product of the units of A and H, while the multiplication is given by



In fact, one can describe this crossed product as the composite algebra of a wreath. The role of t is played by A, the role of s by H, and we define $\lambda : H \otimes A \to A \otimes H$ to be the composite

$$H \otimes A \xrightarrow{\delta \otimes A} H \otimes H \otimes A \xrightarrow{H \otimes c} H \otimes A \otimes H \xrightarrow{\alpha \otimes H} A \otimes H,$$

where c denotes the symmetry; we define $v : H \otimes H \to A \otimes H$ to be

$$H \otimes H \xrightarrow{\delta \otimes \delta} H \otimes H \otimes H \otimes H \xrightarrow{H \otimes c \otimes H} H \otimes H \otimes H \otimes H \xrightarrow{\rho \otimes \mu} A \otimes H$$

and $\sigma : k \to A \otimes H$ to be $\eta \otimes \eta$. The composite is now $A\sharp_\rho H$.

Example 3.4. This time we describe a class of wreaths in Span. Recall that $\mathsf{KL}(\mathsf{Span})$ is given by $\mathsf{kl}(\mathsf{Fam})$, the 2-category whose objects are the categories with coproducts of the form $\mathsf{Fam}(A)$ for a small category A, whose 1-cells are the coproduct-preserving functors, and whose 2-cells are the natural transformations. We have already seen the locally fully faithful 2-functor $\mathsf{Cat} \to \mathsf{kl}(\mathsf{Fam})$; this allows any monad (in Cat) to be seen a monad in $\mathsf{kl}(\mathsf{Fam})$, and so as a wreath in Span.

Thus, the monad (A, t) in Cat becomes a monad $(\mathsf{Fam}(A), \mathsf{Fam}(t))$ in $\mathsf{kl}(\mathsf{Fam})$. Seen as a wreath in Span, this has a composite monad in Span, which will be just a category. Using the description of Kleisli objects in $\mathsf{kl}(\mathsf{Fam})$ given in Section 2.3, one easily sees that this category is just the Kleisli category A_t of (A, t).

Thus, the Kleisli category of a monad in the ordinary sense can be seen as the composite category of a wreath in Span.

Example 3.5. Our final example shows how to consider a category $\mathscr{C}$ with a factorization system $(\mathscr{E}, \mathscr{M})$ as the "composite" of $\mathscr{E}$ and $\mathscr{M}$. Once again this involves a wreath in Span. It builds on the observation of Rosebrugh and Wood [20] that if $\mathscr{C}$ is a category equipped with a *strict* factorization system $(\mathscr{E}, \mathscr{M})$—that is, a factorization system for which every arrow has a *unique* factorization as an arrow in $\mathscr{E}$ followed by an arrow in $\mathscr{M}$—then there is a distributive law between the monads in Span corresponding to the categories $\mathscr{E}$ and $\mathscr{M}$, and that the induced composite monad in Span is the category $\mathscr{C}$. Rosebrugh and Wood suggested one possible treatment of general factorization systems, using a "pseudo" notion of distributive law for 2-categorical monads; here we give an alternative treatment using wreaths.

Let $\mathscr{C}$ be a category equipped with a factorization system $(\mathscr{E}, \mathscr{M})$ in the sense of [5]. Write $\mathscr{C}_0$ for the set of objects of $\mathscr{C}$. For each object x of $\mathscr{C}$, and for each isomorphism class of maps into x lying in $\mathscr{M}$, choose a representative, and write $\mathscr{M}_c$ for the resulting collection of maps in $\mathscr{M}$; we view $\mathscr{M}_c$ as a 1-cell in Span from $\mathscr{C}_0$ to $\mathscr{C}_0$. Note that no compatibility assumptions are made on the choices of representatives.

We shall now define a wreath in Span: playing the role of the monad t is $\mathscr{E}$, and playing the role of s is $\mathscr{M}_c$. The map $\lambda : ts \to st$ arises from the fact that given maps $m : x \to y$ in $\mathscr{M}_c$ and $e : y \to z$ in $\mathscr{E}$, the composite em can be written uniquely as a composite $m'e' : x \to z$ where $e' : x \to y'$ is in $\mathscr{E}$ and $m' : y' \to z$ is in $\mathscr{M}_c$. The map $\sigma : 1 \to st$ is the function which associates to an object x of $\mathscr{C}$ the unique pair (e, m) with $e \in \mathscr{E}$, $m \in \mathscr{M}_c$, and $me = 1_x$. Finally, the map $v : ss \to st$ is the function which associates to maps $m_1 : x \to y$ and $m_2 : y \to z$ in $\mathscr{M}_c$ the unique pair (e, m) with $e \in \mathscr{E}$, $m \in \mathscr{M}_c$, and $me = m_2 m_1$. These data now define a wreath in Span whose composite monad is the category $\mathscr{C}$.

Appendix. Two-dimensional partial maps

In this appendix, we shall construct explicitly a convenient 2-category which is biequivalent to Span. Although a direct proof of this biequivalence is of course possible, we have chosen instead to use an old result of Lawvere [17] concerning "two-dimensional partial maps".

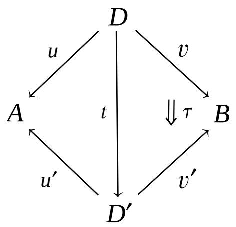
First, however, we recall the "Fam" construction. Given a category B, there is a category $\mathsf{Fam}(B)$, whose objects are the small families of objects of B, and in which an arrow from the family $(b_i)_{i \in I}$ to the family $(c_j)_{j \in J}$ consists of a function $f : I \to J$ and a family $(\beta_i)_{i \in I}$ of morphisms in B, where $\beta_i : b_i \to c_{fi}$.

The resulting category is important for several reasons. First of all, it is the free completion of B under coproducts: there is a 2-category COPR whose objects are the categories with coproducts, whose 1-cells are the coproduct-preserving functors, and whose 2-cells are the natural transformations, and the forgetful 2-functor from COPR to CAT has a left biadjoint whose value at an object B of CAT is $\mathsf{Fam}(B)$, while the unit for the biadjunction is the functor $z : B \to \mathsf{Fam}(B)$ sending an object of B to the singleton family containing that object. This means in particular that for any category C with coproducts, composition with z induces an equivalence of categories between $\mathsf{COPR}(\mathsf{Fam}(B), C)$ and $\mathsf{CAT}(B, C)$.

Secondly, the functor $\pi : \mathsf{Fam}(B) \to \mathsf{Set}$ sending a family of objects to its indexing set is a (split) fibration. The cartesian maps in $\mathsf{Fam}(B)$ are the morphisms

$(f,(\beta_i)_{i\in I}):(b_i)_{i\in I} \to (c_j)_{j\in J}$ in which all the β_i are invertible, while the vertical maps are those for which f is the identity. In fact, it is the morphisms for which the β_i are *identities* which will be of particular interest for us, and we call such morphisms *strictly cartesian*. For any category D, we shall call morphisms in $[D, \mathsf{Fam}(B)]$ strictly cartesian or vertical if all of their components are so. We shall use the fact that every morphism in $[D, \mathsf{Fam}(B)]$ can be written uniquely as a strictly cartesian one followed by a vertical one, and that any vertical map $\omega : zv \to w$ has the form $z\tau : zv \to zs$ for a unique $\tau : v \to s$ (where $z : B \to \mathsf{Fam}(B)$ is defined as in the previous paragraph).

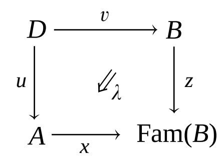
The most important property of $\mathsf{Fam}(B)$ for us, however, is that it classifies "two-dimensional partial maps" into B. If A and B are categories, by a *two-dimensional partial map* from A to B, we mean a category D, and functors $u : D \to A$ and $v : D \to B$ (that is, a span from A to B) with u a discrete opfibration. If (u', v') is another two-dimensional partial map from A to B then a morphism from (u, v) to (u', v') consists of a functor $t : D \to D'$ and a natural transformation $\tau : v \to v't$, as in



With the obvious definition of composition of such morphisms, this gives a category $\mathsf{Par}(\mathsf{Cat})(A, B)$ of two-dimensional partial morphisms from A to B.

Since the discrete opfibrations are stable under pullback and closed under composition, if one composes two-dimensional partial maps as spans, the resulting span will itself be a two-dimensional partial map. One can extend this composition to the morphisms of two-dimensional partial maps and so obtain a bicategory $\mathsf{Par}(\mathsf{Cat})$ whose objects are the small categories, and whose hom-categories are of the form $\mathsf{Par}(\mathsf{Cat})(A, B)$. Rather than verifying this directly, however, we shall approach the bicategorical structure of $\mathsf{Par}(\mathsf{Cat})$ from a different direction.

Given a functor $x : A \to \mathsf{Fam}(B)$, consider the universal diagram



in which λ is strictly cartesian; explicitly, this may be formed as the full subcategory of the comma category z/x consisting of those objects $(b, \phi : zb \to xa, a)$ for which ϕ is strictly cartesian; we call this the *c-comma object* and denote it by $z/_{\mathrm{c}}x$. Then u is a discrete opfibration (because in fact D is the category of elements of $\pi x : A \to \mathsf{Set}$ and $u : D \to A$ the associated discrete opfibration) and so (u, v) is a two-dimensional partial map from A to B. This will turn out to be the value at the object x of an equivalence of categories $S:\mathsf{CAT}(A,\mathsf{Fam}(B)) \to \mathsf{Par}(\mathsf{Cat})(A,B)$. If $x,x':A\to\mathsf{Fam}(B)$ are objects of $\mathsf{CAT}(A,\mathsf{Fam}(B))$, and

$$\begin{array}{ccc} D & \xrightarrow{\;v\;} & B \\ {\scriptstyle u}\downarrow & \Swarrow_{\lambda} & \downarrow{\scriptstyle z} \\ A & \xrightarrow[\;x\;]{} & \mathsf{Fam}(B) \end{array} \qquad\qquad \begin{array}{ccc} D' & \xrightarrow{\;v'\;} & B \\ {\scriptstyle u'}\downarrow & \Swarrow_{\lambda'} & \downarrow{\scriptstyle z} \\ A & \xrightarrow[\;x'\;]{} & \mathsf{Fam}(B) \end{array}$$

are the corresponding c-comma objects, then

(i) to give a morphism $(t,\tau):(u,v)\to(u',v')$ of two-dimensional partial maps is, by the universal property of the c-comma object D', equally to give a functor $s:D\to B$, a strictly cartesian 2-cell $\sigma:zs\to x'u$, and a natural transformation $\tau:v\to s$;
(ii) to give (s,σ,τ) as above is equally to give a functor $w=zs:D\to\mathsf{Fam}(B)$, a strictly cartesian $\sigma:w\to x'u$, and a vertical $\omega=z\tau:zv\to w$;
(iii) to give (w,σ,ω) as above is just to give an arbitrary 2-cell $\zeta:zv\to x'u$; and
(iv) to give ζ as above is equally to give a 2-cell $\xi:x\to x'$ (here $\zeta=\xi u.\lambda$).

Thus, we obtain a bijection between morphisms $Sx\to Sx'$ in $\mathsf{Par}(\mathsf{Cat})(A,B)$ and morphisms $x\to x'$ in $\mathsf{CAT}(A,\mathsf{Fam}(B))$. Since this bijection clearly preserves composition, we obtain a fully faithful functor $S:\mathsf{CAT}(A,\mathsf{Fam}(B))\to\mathsf{Par}(\mathsf{Cat})(A,B)$. To see that S is essentially surjective on objects, and so an equivalence of categories, let

$$\begin{array}{ccc} & D & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ A & & B \end{array}$$

be a two-dimensional partial map from A to B. We define a functor $x:A\to\mathsf{Fam}(B)$ by sending an object a of A to the family $(vd)_{d\in u^{-1}(a)}$ of objects of B, and a morphism $\alpha:a\to a'$ to the morphism $(g,(\bar{\alpha}_d)_{d\in u^{-1}(a)}):(vd)_{d\in u^{-1}(a)}\to(vd')_{d'\in u^{-1}(a')}$ in $\mathsf{Fam}(B)$ defined via the cocartesian liftings

$$\begin{array}{c} d \overset{\bar{\alpha}_d}{\cdots\cdots\to} g(d), \\[2ex] a \xrightarrow{\;\alpha\;} a' \end{array}$$

for each $d\in u^{-1}(a)$. It is straightforward to verify that $S(x)$ is now isomorphic to the original two-dimensional partial map, completing the proof of:

Proposition A.1. *The functor* $S:\mathsf{CAT}(A,\mathsf{Fam}(B))\to\mathsf{Par}(\mathsf{Cat})(A,B)$ *is an equivalence of categories.*

It is this equivalence which expresses what we mean by saying that Fam is a classifier for two-dimensional partial maps.

Now by the fact that $\mathsf{Fam}(A)$ is the free completion of A under coproducts, we deduce an equivalence between $\mathsf{CAT}(A, \mathsf{Fam}(B))$ and $\mathsf{COPR}(\mathsf{Fam}(A), \mathsf{Fam}(B))$, which combined with the previous equivalence gives:

Proposition A.2. *The categories* $\mathsf{COPR}(\mathsf{Fam}(A), \mathsf{Fam}(B))$ *and* $\mathsf{Par}(\mathsf{Cat})(A, B)$ *are equivalent.*

This allows us to make $\mathsf{Par}(\mathsf{Cat})$ into a bicategory biequivalent to the full sub-2-category of COPR consisting of those objects of the form $\mathsf{Fam}(A)$ for a small category A. We call this latter 2-category $\mathsf{kl}(\mathsf{Fam})$.

If A and B are (small) sets, regarded as discrete categories, then $\mathsf{Par}(\mathsf{Cat})(A, B)$ is clearly just $\mathsf{Span}(A, B)$, thus the biequivalence $\mathsf{Par}(\mathsf{Cat}) \sim \mathsf{kl}(\mathsf{Fam})$ restricts to a biequivalence between Span and the full sub-2-category $\mathsf{kl}(\mathsf{Fam})_{\mathrm{D}}$ of $\mathsf{kl}(\mathsf{Fam})$ consisting of those objects of the form $\mathsf{Fam}(A)$ for a small *discrete* A.

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