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# Abstract substitution in enriched categories

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## Abstract

Enriched categories equipped with an abstract substitution process are defined in this paper, and called *substitudes*. They generalize both monoidal enriched categories and operads, and are a little more general than multicategories. They can bear braidings and symmetries. There are two convolution processes with distinctly different properties.

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## 0. Introduction

Substitudes slightly generalize multicategories in the sense of Lambek (see [10,15]). In [7] we called them lax procomonoidal categories as one of the many related structures. Our purpose here is to centre attention on substitudes in the enriched context and to further study two convolution constructions which we now call standard and non-standard. The standard convolution reduces to the original one defined by the first author in [3] when the substitudes are (pro)monoidal where it was pointed out that the monoids are the (pro)monoidal functors; this generalizes to the present setting. The non-standard convolution was inspired by Bakalov et al. [1] as was our treatment here of the “annihilator algebra”.

Substitudes generalize both monoidal categories and operads. It is therefore natural to define braided and symmetric substitudes which generalize braided and symmetric monoidal categories in the sense of Eilenberg and Kelly [8] (and Joyal and Street [13])

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and the symmetric operads of Peter May [16]. We define commutative monoids in braided substitutes and Lie algebras in additive braided substitutes. Each substitute can be symmetrized in a simple way that has no analogue for monoidal categories.

We deal throughout with enriched categories for which the basic reference is Kelly's book [14].

## 1. Substitutes

Let  $\mathcal{V}$  denote a symmetric closed monoidal category which is both complete and cocomplete. We often write as if  $\mathcal{V}$  were strictly associative and unital.

A  $\mathcal{V}$ -substitute is a  $\mathcal{V}$ -category  $\mathcal{A}$  together with:

- for each integer  $n \geq 0$ , a  $\mathcal{V}$ -functor

$$P_n: \underbrace{\mathcal{A}^{\text{op}} \otimes \cdots \otimes \mathcal{A}^{\text{op}}}_n \otimes \mathcal{A} \rightarrow \mathcal{V}$$

whose value at  $(A_1, \dots, A_n, A)$  is denoted by  $P_n(A_1, \dots, A_n; A)$ ;

- for each partition<sup>1</sup>  $\xi: m_1 + \cdots + m_n = m$ , a  $\mathcal{V}$ -natural family of morphisms

$$\mu_\xi: P_n(A_\bullet; A) \otimes P_{m_1}(A_{1\bullet}; A_1) \otimes \cdots \otimes P_{m_n}(A_{n\bullet}; A_n) \rightarrow P_\xi(A_{\bullet\bullet}; A)$$

in  $\mathcal{V}$ , called *substitution*, where we use the shorthand

$$P_k(X_\bullet; X) = P_k(X_1, \dots, X_k; X)$$

and

$$P_\xi(X_{\bullet\bullet}; X) = P_m(X_{11}, \dots, X_{1m_1}, \dots, X_{n1}, \dots, X_{nm_n}; X);$$

and

- a  $\mathcal{V}$ -natural family of morphisms

$$\eta: \mathcal{A}(A, B) \rightarrow P_1(A; B)$$

in  $\mathcal{V}$ , called the *unit*;

subject to the commutativity of the following three diagrams involving the partitions  $\xi: m_{11} + \cdots + m_{1n_1} + \cdots + m_{r1} + \cdots + m_{rn_r} = m$ ,  $\xi_i: m_{i1} + \cdots + m_{in_i} = m_i$ ,  $\zeta: n_1 + \cdots + n_r = n$ ,  $\zeta\xi: m_1 + \cdots + m_r = m$ ,  $1_n: 1 + \cdots + 1 = n$  and  $!: m = m$ , and making use of a slight extension of the shorthand.

<sup>1</sup> The summands in our partitions are allowed to be zero and in non-monotone order.

$$\begin{array}{ccc}
& P_r(A_\bullet; A) \otimes \bigotimes_{i=1}^r P_{n_i}(A_i, \bullet; A_i) \otimes \bigotimes_{i=1}^r \bigotimes_{j_i=1}^{n_i} P_{m_{ij_i}}(A_{ij_i}, \bullet; A_{ij_i}) \\
& \swarrow \cong \\
P_r(A_\bullet; A) \otimes \bigotimes_{i=1}^r \left( P_{n_i}(A_i, \bullet; A_i) \otimes \bigotimes_{j_i=1}^{n_i} P_{m_{ij_i}}(A_{ij_i}, \bullet; A_{ij_i}) \right) & & P_\zeta(A_{\bullet\bullet}; A) \otimes \bigotimes_{i=1}^r \bigotimes_{j_i=1}^{n_i} P_{m_{ij_i}}(A_{ij_i}, \bullet; A_{ij_i}) \\
& \downarrow 1 \otimes \bigotimes_{i=1}^r \mu_{\zeta_i} & \searrow \mu_\zeta \otimes 1 \\
P_r(A_\bullet; A) \otimes \bigotimes_{i=1}^r P_{\zeta_i}(A_i, \bullet\bullet; A_i) & \xrightarrow{\mu_{\zeta_\zeta}} & P_{\zeta, \zeta}(A_{\bullet\bullet\bullet}; A)
\end{array}$$
  

$$\begin{array}{ccc}
P_n(A_\bullet; A) \otimes \bigotimes_{i=1}^n \mathcal{A}(B_i; A_i) & \xrightarrow{\text{action}} & \mathcal{A}(A, B) \otimes P_m(A_\bullet; A) \xrightarrow{\eta \otimes 1} P_1(A; B) \otimes P_m(A_\bullet; A) \\
& \downarrow 1 \otimes \bigotimes_{i=1}^n \eta & \searrow \text{action} \quad \swarrow \mu_1 \\
P_n(A_\bullet; A) \otimes \bigotimes_{i=1}^n P_1(B_i; A_i) & \xrightarrow{\mu_{1_n}} & P_m(A_\bullet; B)
\end{array}$$

A  $\mathcal{V}$ -substitute is what was called a “lax procomonoidal  $\mathcal{V}$ -category” in [7]; it (or, more pedantically, a small one) is precisely a lax monoid in  $(\mathcal{V}\text{-}\mathbf{Mod})^{\text{op}}$ . A *pre-monoidal  $\mathcal{V}$ -category* in the sense of Day [4], later called *promonoidal*, is essentially a substitute in which the morphisms  $\mu_\xi$  are universal (that is, express their codomains as coends over  $A_1, \dots, A_n$  of their domains) and the morphisms  $\eta$  are invertible.

An *oplax monoidal  $\mathcal{V}$ -category* is a  $\mathcal{V}$ -substitute for which each of the  $\mathcal{V}$ -functors

$$P_n(A_1, \dots, A_n; -): \mathcal{A} \rightarrow \mathcal{V}$$

is representable. This agrees with the terminology of Day and Street [7] and amounts to a lax monoid in  $(\mathcal{V}\text{-}\mathbf{Cat})^{\text{co}}$ . If we denote the representing object of the above displayed  $\mathcal{V}$ -functor by  $\bigotimes_n(A_1, \dots, A_n) \in \mathcal{A}$  then we obtain functors

$$\bigotimes_n: \underbrace{\mathcal{A} \otimes \dots \otimes \mathcal{A}}_n \rightarrow \mathcal{A}$$

and isomorphisms

$$P_n(A_1, \dots, A_n; A) \cong \mathcal{A} \left( \bigotimes_n(A_1, \dots, A_n), A \right),$$

$\mathcal{V}$ -natural in all the variables  $A_1, \dots, A_n$  and  $A$ . Notice that, by the Yoneda lemma, the substitution  $\mu_\xi$  induces a *cosubstitution*

$$\delta_\xi: \bigotimes_m(A_{11}, \dots, A_{nm_n}) \rightarrow \bigotimes_n \left( \bigotimes_{m_1}(A_{11}, \dots, A_{1m_1}), \dots, \bigotimes_{m_n}(A_{n1}, \dots, A_{nm_n}) \right)$$

and the unit  $\eta$  induces a *counit*  $\varepsilon: \bigotimes_1(A) \rightarrow A$  satisfying the appropriate three conditions. A *monoidal  $\mathcal{V}$ -category* is essentially an oplax monoidal  $\mathcal{V}$ -category in which all the  $\delta_\xi$  and  $\varepsilon$  are invertible.

A *lax monoidal  $\mathcal{V}$ -category* is a lax monoid in  $\mathcal{V}\text{-Cat}$ . This means that a lax monoid structure on a  $\mathcal{V}$ -category  $\mathcal{A}$  amounts precisely to an oplax monoidal structure on  $\mathcal{A}^{\text{op}}$ . A monoidal  $\mathcal{V}$ -category is also essentially a lax monoidal  $\mathcal{V}$ -category in which all the  $\mu_\xi$  and  $\eta$  are invertible.

A lax monoidal  $\mathcal{V}$ -category  $\mathcal{X}$  is called *cocomplete* when it is cocomplete as a  $\mathcal{V}$ -category and each of the functors  $\otimes_n$  preserves colimits in each of the  $n$  variables.

Suppose  $\mathcal{A}$  and  $\mathcal{X}$  are  $\mathcal{V}$ -substitutes. A  $\mathcal{V}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{X}$  is called a *substitute morphism* when it is equipped with, for each integer  $n \geq 0$ , a family of morphisms

$$\phi_n: P_n(A_1, \dots, A_n; A) \rightarrow P_n(FA_1, \dots, FA_n; FA),$$

$\mathcal{V}$ -natural in all the variables  $A_1, \dots, A_n$  and  $A$ , satisfying the conditions

$$\phi_m \circ \mu_\xi = \mu_\xi \circ \left( \phi_n \otimes \bigotimes_{r=1}^n \phi_{m_r} \right) \quad \text{and} \quad \phi_1 \circ \eta = \eta \circ F_{A,B}$$

of compatibility with substitution and unit. The morphisms  $\phi_n$  induce their mates

$$\rho_n: \int^A P_n(A_1, \dots, A_n; A) \otimes \mathcal{X}(FA, X) \rightarrow P_n(FA_1, \dots, FA_n; X).$$

The substitute morphism  $F$  is called *strong* when each  $\rho_n$  is invertible. *Oplax monoidal  $\mathcal{V}$ -category morphisms* are substitute morphisms; we identify  $\phi_n$  with the morphism induced (Yoneda) between the representing multiple tensor products. *Lax monoidal  $\mathcal{V}$ -category morphisms* are defined dually.

A morphism between two  $\mathcal{V}$ -substitute structures on the same  $\mathcal{V}$ -category  $\mathcal{A}$  is a substitute morphism structure on the identity  $\mathcal{V}$ -functor of  $\mathcal{A}$ . Similarly one defines morphisms of (op)lax monoidal structures on  $\mathcal{A}$ .

Suppose  $F$  and  $G: \mathcal{A} \rightarrow \mathcal{X}$  are substitute morphisms. A *substitute transformation*  $\theta: F \rightarrow G$  is a natural transformation such that the following commutes:

$$\begin{array}{ccc} P_n(A_1, \dots, A_n; A) & \xrightarrow{\phi_n} & P_n(FA_1, \dots, FA_n; FA) \\ \phi_n \downarrow & & \downarrow P_n(1, \dots, 1; \theta_A) \\ P_n(GA_1, \dots, GA_n; GA) & \xrightarrow{P_n(\theta_{A_1}, \dots, \theta_{A_n}; 1)} & P_n(FA_1, \dots, FA_n; GA). \end{array}$$

Write  $\mathbf{Sstd}(\mathcal{A}, \mathcal{X})$  for the category of substitute morphisms and transformations.

In any monoidal bicategory it is possible to define a notion of “representation” of a lax comonoid in a lax monoid. Here we are concerned only with the case of a *representation*  $F: \mathcal{A} \rightarrow \mathcal{X}$  of a  $\mathcal{V}$ -substitute  $\mathcal{A}$  in a cocomplete lax monoidal  $\mathcal{V}$ -category  $\mathcal{X}$ ; it is a  $\mathcal{V}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{X}$  together with a  $\mathcal{V}$ -natural family of morphisms

$$\phi_n: P_n(A_1, \dots, A_n; A) \otimes_n (FA_1, \dots, FA_n) \rightarrow FA$$

in  $\mathcal{X}$  such that the following two diagrams commute.

$$\begin{array}{c}
P_n(A_\bullet; A) \otimes P_{m_1}(A_1 \bullet; A_1) \otimes \dots \otimes P_{m_n}(A_n \bullet; A_n) \otimes \otimes_n (\otimes_{m_1}(FA_1 \bullet), \dots, \otimes_{m_n}(FA_n \bullet)) \\
\swarrow \mu_\xi \otimes \mu_\xi \quad \searrow \text{canon} \\
P_n(A_\bullet; A) \otimes \otimes_n \left( P_{m_1}(A_1 \bullet; A_1) \otimes \otimes_{m_1}(FA_1 \bullet), \dots, P_{m_n}(A_n \bullet; A_n) \otimes \otimes_{m_n}(FA_n \bullet) \right) \\
\downarrow 1 \otimes \otimes_n (\phi_{m_1}, \dots, \phi_{m_n}) \\
P_m(A_1 \bullet, \dots, A_n \bullet; A) \otimes \otimes_m (FA_1 \bullet, \dots, FA_n \bullet) \quad P_n(A_\bullet; A) \otimes \otimes_n (FA_\bullet) \\
\swarrow \phi_m \quad \searrow \phi_n \\
FA
\end{array}$$
  

$$\begin{array}{ccc}
\mathcal{A}(A_1, A) \otimes FA_1 & \xrightarrow{\text{action}} & FA \\
\eta \otimes \eta \searrow & & \nearrow \phi_1 \\
& P_1(A_1; A) \otimes \otimes_1 (FA_1) &
\end{array}$$

In the special case when  $\mathcal{X}$  is monoidal, so that  $\mathcal{X}$  is both a substitute and lax monoidal, a representation  $F: \mathcal{A} \rightarrow \mathcal{X}$  is precisely a substitute morphism.

By way of example, we conclude this section with some remarks about operads. We believe that it is more convenient to use the term *operad* for what is commonly called a “non-permutative operad”: the original operads of May [16] can then be called “symmetric operads”. Accordingly, we define an *operad*  $T$  in  $\mathcal{V}$  to be a  $\mathcal{V}$ -substitute  $\Sigma T$  whose underlying category is the  $\mathcal{V}$ -category  $I$  with one object  $*$  and with hom  $I(*, *)$  equal to the unit  $I$  for tensor in  $\mathcal{V}$ . One writes  $T_n$  for the object  $P_n(*, \dots, *, *)$  of  $\mathcal{V}$ . We call the substitute  $\Sigma T$  the *suspension* of the operad  $T$ . A  $T$ -*algebra* in a  $\mathcal{V}$ -substitute  $\mathcal{A}$  is defined to be a substitute morphism  $A: \Sigma T \rightarrow \mathcal{A}$ . On the other hand, we define a  $T$ -*algebra* in a lax monoidal category  $\mathcal{X}$  to be a representation of  $\Sigma T$  in  $\mathcal{X}$ .

In particular, there is an operad which we identify with the unit  $I$  in  $\mathcal{V}$ ; we have  $I_n = I$ . A *monoid* in a  $\mathcal{V}$ -substitute or in a lax monoidal  $\mathcal{V}$ -category is defined to be an  $I$ -algebra.

## 2. Standard convolution

We write  $[\mathcal{A}, \mathcal{X}]$  for the  $\mathcal{V}$ -category of  $\mathcal{V}$ -functors from a  $\mathcal{V}$ -category  $\mathcal{A}$  to a  $\mathcal{V}$ -category  $\mathcal{X}$  (see [14], for example).

Suppose  $\mathcal{A}$  is a small  $\mathcal{V}$ -substitute and  $\mathcal{X}$  is a cocomplete lax monoidal  $\mathcal{V}$ -category. For  $\mathcal{V}$ -functors  $F_1, \dots, F_n: \mathcal{A} \rightarrow \mathcal{X}$ , we define a  $\mathcal{V}$ -functor  $\otimes_n(F_1, \dots, F_n): \mathcal{A} \rightarrow \mathcal{X}$  by

$$\otimes_n(F_1, \dots, F_n) = \int^{A_1, \dots, A_n} P_n(A_1, \dots, A_n; -) \otimes \otimes_n (F_1 A_1, \dots, F_n A_n).$$

A substitution morphism  $\mu_{\xi}^{\zeta}: \otimes_n (\otimes_{m_1}(F_{1\bullet}), \dots, \otimes_{m_n}(F_{n\bullet})) \rightarrow \otimes_m(F_{1\bullet}, \dots, F_{n\bullet})$  is defined by the composite

$$\begin{aligned} & \int^{A_{\bullet}} P_n(A_{\bullet}; -) \otimes \otimes_n \left( \int^{A_{1\bullet}} P_{m_1}(A_{1\bullet}; A_1) \otimes F_{1\bullet} A_{1\bullet}, \dots, \int^{A_{n\bullet}} P_{m_n}(A_{n\bullet}; A_n) \otimes F_{n\bullet} A_{n\bullet} \right) \\ & \xrightarrow{\cong} \int^{A_{\bullet}, A_{1\bullet}, \dots, A_{n\bullet}} P_n(A_{\bullet}; -) \otimes P_{m_1}(A_{1\bullet}; A_1) \otimes \dots \otimes P_{m_n}(A_{n\bullet}; A_n) \\ & \quad \otimes \otimes_n \left( \otimes_{m_1}(F_{1\bullet} A_{1\bullet}), \dots, \otimes_{m_n}(F_{n\bullet} A_{n\bullet}) \right) \xrightarrow{\int^{A_{1\bullet}, \dots, A_{n\bullet}} \mu_{\xi}^{\zeta} \otimes \mu_{\xi}^{\zeta}} \\ & \quad \int^{A_{1\bullet}, \dots, A_{n\bullet}} P_m(A_{1\bullet}, \dots, A_{n\bullet}; -) \otimes \otimes_m(F_{1\bullet} A_{1\bullet}, \dots, F_{n\bullet} A_{n\bullet}) \end{aligned}$$

in which the isomorphism comes from the cocontinuity in each variable of the  $\mathcal{V}$ -functor  $\otimes_n$  of  $\mathcal{X}$ . A unit  $\eta: F_1 \rightarrow \otimes_1(F_1)$  is defined by the composite

$$F_1 \xrightarrow{\cong} \int^{A_1} \mathcal{A}(A_1, -) \otimes F_1 A_1 \xrightarrow{\int^{A_1} \eta^{\otimes} \eta} \int^{A_1} P_1(A_1; -) \otimes \otimes_1(F_1 A_1)$$

in which the isomorphism is that of Yoneda's lemma.

**Proposition 2.1.**  *$[\mathcal{A}, \mathcal{X}]$  becomes a cocomplete lax monoidal  $\mathcal{V}$ -category using the above substitution and unit.*

We shall call this the *standard convolution* lax monoidal structure on  $[\mathcal{A}, \mathcal{X}]$ . The case where  $\mathcal{X} = \mathcal{V}$  was the first convolution structure defined in [7, Section 7]. That case has further properties.

**Proposition 2.2.** *Standard convolution defines an equivalence between the category of substitute structures on any small  $\mathcal{V}$ -category  $\mathcal{A}$  and the category of cocomplete lax monoidal structures on  $[\mathcal{A}, \mathcal{V}]$ .*

For cocomplete lax monoidal  $\mathcal{V}$ -categories  $\mathcal{Z}$  and  $\mathcal{X}$ , write  $\mathbf{CocLM}(\mathcal{Z}, \mathcal{X})$  for the category of morphisms of lax monoidal  $\mathcal{V}$ -categories whose underlying  $\mathcal{V}$ -functors preserve colimits (that is, are “cocontinuous”). The following result should be compared with [11]; we refrain from stating the several variable form which also holds.

**Proposition 2.3.** *For any substitute  $\mathcal{A}$ , the Yoneda embedding  $\mathbf{Y}^{\dagger}: \mathcal{A} \rightarrow [\mathcal{A}, \mathcal{V}]^{\text{op}}$  is a substitute morphism where  $[\mathcal{A}, \mathcal{V}]^{\text{op}}$  is the substitute represented by the standard convolution structure on  $[\mathcal{A}, \mathcal{V}]$ . Restriction along  $\mathbf{Y}^{\dagger}$  is an equivalence of*

categories

$$\mathbf{Sstd}(\mathcal{A}, \mathcal{X}^{\text{op}}) \simeq \mathbf{CocLM}([\mathcal{A}, \mathcal{V}], \mathcal{X})^{\text{op}}.$$

For substitutes  $\mathcal{A}$  and  $\mathcal{B}$  there is an obvious substitute structure on the  $\mathcal{V}$ -category  $\mathcal{A} \otimes \mathcal{B}$ ; take

$$P_n((A_1, B_1), \dots, (A_n, B_n); (A, B)) = P_n(A_1, \dots, A_n; A) \otimes P_n(B_1, \dots, B_n; B)$$

and so on.

**Proposition 2.4.** *For  $\mathcal{V}$ -substitutes  $\mathcal{A}$  and  $\mathcal{B}$ , and a cocomplete lax monoidal  $\mathcal{V}$ -category  $\mathcal{X}$ , the canonical isomorphism*

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{X}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{X}]]$$

*of  $\mathcal{V}$ -categories is a morphism of lax monoidal  $\mathcal{V}$ -categories where all three  $\mathcal{V}$ -functor  $\mathcal{V}$ -categories have the standard convolution structures.*

Generalizing the observation of [3, Section 3.1, p. 72], we deduce from the last proposition that a monoid in  $[\mathcal{A}, \mathcal{X}]$  is precisely a substitute morphism from  $\mathcal{A}$  to  $\mathcal{X}$  as defined in Section 1. Hence, for any  $\mathcal{V}$ -operad  $T$ , the  $\mathcal{V}$ -category of  $T$ -algebras in  $\mathcal{X}$  is the  $\mathcal{V}$ -category of monoids in the standard convolution  $[\Sigma T, \mathcal{X}]$ .

The final proposition of this section is a lax version of a result of Day and Street [5].

**Proposition 2.5.** *Suppose  $J: \mathcal{A} \rightarrow \mathcal{H}$  is a strong substitute morphism with  $\mathcal{A}$  small. Suppose  $\mathcal{X}$  is a cocomplete lax monoidal  $\mathcal{V}$ -category. Then the  $\mathcal{V}$ -functor*

$$\exists_J: [\mathcal{A}, \mathcal{X}] \rightarrow [\mathcal{H}, \mathcal{X}],$$

*given by left Kan extension along  $J$ , is a strong morphism of lax monoidal  $\mathcal{V}$ -categories.*

### 3. Non-standard convolution

For any small  $\mathcal{V}$ -substitute  $\mathcal{A}$  there is a  $\mathcal{V}$ -substitute structure on the  $\mathcal{V}$ -category  $[\mathcal{A}^{\text{op}}, \mathcal{V}]$  which we described at the end of Day and Street [7] in order to include the fundamental “pseudo-tensor category” used in [1]. For  $\mathcal{V}$ -functors  $F_1, \dots, F_n, F: \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}$ , we put

$$\begin{aligned} P_n(F_1, \dots, F_n; F) &= \int_{A_1, \dots, A_n} \left[ F_1 A_1 \otimes \dots \otimes F_n A_n, \int^A P_n(A_1, \dots, A_n; A) \otimes F A \right] \\ &= [\mathcal{A}^{\otimes n \text{ op}}, \mathcal{V}] \left[ F_1 \otimes \dots \otimes F_n, \int^A P_n(-; A) \otimes F A \right]. \end{aligned}$$

The substitution  $\mu_{\xi} : P_n(F_{\bullet}; F) \otimes P_{m_1}(F_{1\bullet}; F_1) \otimes \cdots \otimes P_{m_n}(F_{n\bullet}; F_n) \rightarrow P_{\xi}(F_{\bullet\bullet}; F)$  is defined by the following composite:

$$\begin{aligned}
& [\mathcal{A}^{\otimes n \text{ op}}, \mathcal{V}] \left[ \bigotimes_n F_{\bullet}, \int^A P_n(-; A) \otimes FA \right] \\
& \quad \bigotimes_{r=1}^n [\mathcal{A}^{\otimes m_r \text{ op}}, \mathcal{V}] \left[ \bigotimes_{m_r} F_{\bullet}, \int^{A_r} P_{m_r}(-; A_r) \otimes F_r A_r \right] \\
& \xrightarrow{1 \otimes \bigotimes_{r=1}^n} [\mathcal{A}^{\otimes n \text{ op}}, \mathcal{V}] \left[ \bigotimes_n F_{\bullet}, \int^A P_n(-; A) \otimes FA \right] \\
& \quad \otimes [\mathcal{A}^{\otimes m \text{ op}}, \mathcal{V}] \left[ \bigotimes_{\xi} F_{\bullet\bullet}, \int^{A_{\bullet}} \bigotimes_{r=1}^n P_{m_r}(-; A_r) \otimes \bigotimes_n F_{\bullet} A_{\bullet} \right] \\
& \xrightarrow{\text{comp}} [\mathcal{A}^{\otimes m \text{ op}}, \mathcal{V}] \left[ \bigotimes_{\xi} F_{\bullet\bullet}, \int^{A_{\bullet}} \bigotimes_{r=1}^n P_{m_r}(-; A_r) \otimes \int^A P_n(A_{\bullet}; A) \otimes FA \right] \\
& \xrightarrow{[1, f^A \mu_{\xi} \otimes 1]} [\mathcal{A}^{\otimes m \text{ op}}, \mathcal{V}] \left[ \bigotimes_{\xi} F_{\bullet\bullet}, \int^A P_{\xi}(A_{\bullet\bullet}; A) \otimes FA \right],
\end{aligned}$$

while the unit  $\eta : [\mathcal{A}^{\text{op}}, \mathcal{V}](F_1, F) \rightarrow P_1(F_1; F)$  is the following composite:

$$\begin{aligned}
& [\mathcal{A}^{\text{op}}, \mathcal{V}](F_1, F) \cong [\mathcal{A}^{\text{op}}, \mathcal{V}] \left[ F_1, \int^A \mathcal{A}(-, A) \otimes FA \right] \\
& \xrightarrow{[1, f^A \eta \otimes 1]} [\mathcal{A}^{\text{op}}, \mathcal{V}] \left[ F_1, \int^A P_1(-; A) \otimes FA \right].
\end{aligned}$$

**Proposition 3.1.** *For any small substitute  $\mathcal{A}$  the Yoneda embedding  $\mathbf{Y} : \mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathcal{V}]$  is a strong substitute morphism into the non-standard convolution substitute.*

**Proof.** Using the Yoneda lemma, we have

$$\begin{aligned}
& \int^A P_n(A_{\bullet}; A) \otimes [\mathcal{A}^{\text{op}}, \mathcal{V}](\mathbf{Y}A, F) \cong \int^A P_n(A_{\bullet}; A) \otimes FA \\
& \cong [\mathcal{A}^{\otimes n \text{ op}}, \mathcal{V}] \left( \bigotimes_n \mathbf{Y}A_{\bullet}, \int^A P_n(-; A) \otimes FA \right) \cong P_n(\mathbf{Y}A_{\bullet}; F). \quad \square
\end{aligned}$$

By combining Propositions 2.5 and 3.1, we obtain the next result.



**Corollary 3.2.** *For any small substitute  $\mathcal{A}$  and any cocomplete lax monoidal  $\mathcal{V}$ -category  $\mathcal{X}$ , left Kan extension along the Yoneda embedding  $\mathbf{Y}: \mathcal{A} \rightarrow [\mathcal{A}^{\text{op}}, \mathcal{V}]$  is a strong morphism*

$$\exists_{\mathbf{Y}}: [\mathcal{A}, \mathcal{X}] \rightarrow [[\mathcal{A}^{\text{op}}, \mathcal{V}], \mathcal{X}]$$

*of lax monoidal  $\mathcal{V}$ -categories.*

Let us define the  $K$ -annihilator of  $F$  by

$$\mathbf{Ann}_K F = \exists_{\mathbf{Y}}(K)(F) = \int^{\mathcal{A}} FA \otimes KA.$$

Corollary 3.2 explains the general mechanics behind the “annihilation algebra” of [1, Corollary 7.1, p. 45]. The case of interest in that paper is where  $\mathcal{X} = [\mathcal{A}, \mathcal{V}]$  with the standard convolution structure, and  $K$  is a monoid in  $[\mathcal{A}, \mathcal{X}] \cong [\mathcal{A} \otimes \mathcal{A}, \mathcal{V}]$ ; in other words,  $K$  is a substitute morphism  $K: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{V}$ . Then  $\mathbf{Ann}_K: [\mathcal{A}^{\text{op}}, \mathcal{V}] \rightarrow [\mathcal{A}, \mathcal{V}]$  is a morphism between the non-standard and the standard structures.

#### 4. Braided and symmetric substitutes

We begin by reviewing a few facts about some categories that involve braids. We write  $\mathbf{B}$  for the monoidal category of braids [13]; the objects are the natural numbers, all morphisms are automorphisms, the hom  $\mathbf{B}(n, n)$  is the Artin braid group  $\mathbf{B}_n$  on  $n$  strings, and the tensor product is defined to be addition of natural numbers and horizontal stacking of braids. We write  $\mathbf{P}$  for the monoidal category defined similarly using the permutation groups  $\mathbf{P}_n$  rather than the braid groups. Each braid  $\beta$  on  $n$  strings has an underlying permutation  $\beta: i \mapsto \beta(i)$  of the set  $\mathbf{n} = \{1, \dots, n\}$ ; this defines a braided strict monoidal functor  $\mathbf{B} \rightarrow \mathbf{P}$  which is the identity on objects and full. For each finite sequence  $(A_1, \dots, A_n)$  of objects in any braided monoidal category  $\mathcal{V}$ , we obtain a canonical isomorphism  $c_\beta: A_{\beta(1)} \otimes \dots \otimes A_{\beta(n)} \rightarrow A_1 \otimes \dots \otimes A_n$  which is defined to be

$$1_{A_1} \otimes \dots \otimes 1_{A_{i-1}} \otimes c_{A_i, A_{i+1}} \otimes 1_{A_{i+2}} \otimes \dots \otimes 1_{A_n}$$

when  $\beta$  is the braid that crosses the  $i$ th string over the  $(i+1)$ th string, the identity when  $\beta$  is the identity braid, and  $c_\beta = c_{\beta'} \circ c_{\beta''}$  when  $\beta = \beta'' \circ \beta'$ .

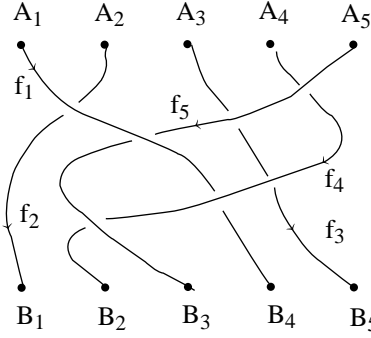
The algebraic simplicial category  $\Delta$  has objects the natural numbers and has hom  $\Delta(m, n)$  consisting of the order-preserving functions  $\xi: \mathbf{m} \rightarrow \mathbf{n}$ ; composition is that of functions. Moreover,  $\Delta$  becomes monoidal under addition of natural numbers and ordinal sum of order-preserving functions.

With respect to strict monoidal functors,  $\mathbf{B}$  is the free braided strict monoidal category generated by a single object,  $\mathbf{P}$  is the free symmetric strict monoidal category generated by a single object, and  $\Delta$  is the free strict monoidal category containing a monoid. First we need to recall from [12, 13] the description of the free braided strict monoidal category  $\mathbf{B} \int \mathcal{A}$  on any category  $\mathcal{A}$ . The objects of  $\mathbf{B} \int \mathcal{A}$  are finite sequences  $(A_1, \dots, A_n)$  of objects of  $\mathcal{A}$ . Morphisms only occur between sequences of

the same length; a morphism

$$(\alpha; f_1, \dots, f_n) : (A_1, \dots, A_n) \rightarrow (B_1, \dots, B_n)$$

consists of  $\alpha \in \mathbf{B}_n$  and  $f_i : A_i \rightarrow B_{\alpha(i)}$  in  $\mathcal{A}$ . Such a morphism can be viewed as the braid  $\alpha$  labelled by  $f_1, \dots, f_n$  as, for example:



Composition of labelled braids is performed by composing the labels on each string of the composite braid. The operation of addition of braids extends in the obvious way to labelled braids yielding a tensor structure on  $\mathbf{B} \int \mathcal{A}$ . There is an obvious braiding on  $\mathbf{B} \int \mathcal{A}$  obtained from the braiding on  $\mathbf{B}$  by labelling the strings with identity morphisms. We have an inclusion functor  $i : \mathcal{A} \rightarrow \mathbf{B} \int \mathcal{A}$  identifying  $\mathcal{A}$  with the labelled braids with a single string.

If  $\mathcal{V}$  is braided monoidal, there is a “substitution” operation  $\otimes : \mathbf{B} \int \mathcal{V} \rightarrow \mathcal{V}$  whose composite with  $i : \mathcal{V} \rightarrow \mathbf{B} \int \mathcal{V}$  is isomorphic to the identity functor; explicitly, it takes  $(A_1, \dots, A_n)$  to  $\otimes_n (A_1, \dots, A_n) = A_1 \otimes \dots \otimes A_n$  (bracketed from the left, say) and takes  $(\alpha; f_1, \dots, f_n)$  to the morphism

$$\alpha[f_1, \dots, f_n] : A_1 \otimes \dots \otimes A_n \rightarrow B_1 \otimes \dots \otimes B_n$$

obtained by composing  $f_1 \otimes \dots \otimes f_n$  with the canonical isomorphism

$$c_\alpha : B_{\alpha(1)} \otimes \dots \otimes B_{\alpha(n)} \rightarrow B_1 \otimes \dots \otimes B_n.$$

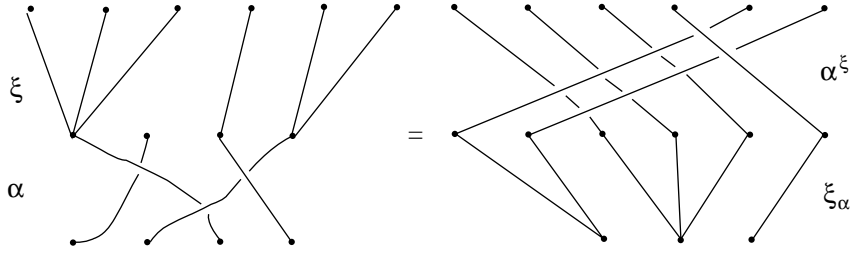
In particular, we are interested in the case  $\mathcal{V} = \mathbf{B}$ . We can identify an object  $(m_1, \dots, m_n)$  of  $\mathbf{B} \int \mathbf{B}$  with the order-preserving function  $\xi : \mathbf{m} \rightarrow \mathbf{n}$  whose fibre over  $i \in \mathbf{n}$  has cardinality  $m_i$ . For any braid  $\alpha \in \mathbf{B}_n$  we write  $\xi_\alpha : \mathbf{m} \rightarrow \mathbf{n}$  for the order-preserving function determined by the object  $(m_{\alpha(1)}, \dots, m_{\alpha(n)})$  of  $\mathbf{B} \int \mathbf{B}$ . Then we have the morphism

$$\alpha^\xi = \alpha[1_{m_1}, \dots, 1_{m_n}] : m \rightarrow m$$

in  $\mathbf{B}$ . In other words, we have defined a function

$$\Delta(m, n) \times \mathbf{B}(n, n) \rightarrow \mathbf{B}(m, m) \times \Delta(m, n)$$

that takes the pair  $(\xi, \alpha)$  to the pair  $(\alpha^\xi, \xi_\alpha)$ ; we call this the *distributive law* for  $\Delta$  over  $\mathbf{B}$ . We can visualize this process as follows.



There is also a distributive law

$$\Delta(m, n) \times \mathbf{P}(n, n) \rightarrow \mathbf{P}(m, m) \times \Delta(m, n)$$

for  $\Delta$  over  $\mathbf{P}$  which we can obtain similarly, or from the braid case by means of the quotient homomorphisms  $\mathbf{B}_n \rightarrow \mathbf{P}_n$  and  $\mathbf{B}_m \rightarrow \mathbf{P}_m$ .

We can use this discussion to define monoidal categories  $\mathbf{B}\Delta$  and  $\mathbf{P}\Delta$  which are the free braided and symmetric (respectively) strict monoidal categories containing a monoid. In other terminology,  $\mathbf{P}\Delta$  is the PROP for monoids (a description occurs in [17, Construction 4.1, p. 215] although there is a misprint<sup>2</sup> in which the outside product should be a sum). To be explicit, the objects of  $\mathbf{B}\Delta$  and  $\mathbf{P}\Delta$  are the natural numbers. A morphism  $(\alpha, \xi): m \rightarrow n$  in  $\mathbf{B}\Delta$  consists of a braid  $\alpha \in \mathbf{B}_m$  and an order-preserving function  $\xi: \mathbf{m} \rightarrow \mathbf{n}$ . The composite of  $(\alpha, \xi): m \rightarrow n$  and  $(\beta, \zeta): n \rightarrow p$  is defined to be  $(\beta^\xi \circ \alpha, \zeta \circ \xi_\beta): m \rightarrow p$ . There is a faithful functor  $\mathbf{B} \rightarrow \mathbf{B}\Delta$  taking  $\alpha: m \rightarrow m$  to  $(\alpha, 1_m): m \rightarrow m$ , and a faithful functor  $\Delta \rightarrow \mathbf{B}\Delta$  taking  $\xi: m \rightarrow n$  to  $(1_m, \xi): m \rightarrow n$ ; both functors are the identity on objects and we identify morphisms in  $\mathbf{B}$  and  $\Delta$  with their image in  $\mathbf{B}\Delta$ . So we may write  $(\alpha, \xi) = \xi \circ \alpha$  and note that the morphisms of  $\mathbf{B}$  are precisely the invertible morphisms of  $\mathbf{B}\Delta$ . There is a braided strict monoidal structure on  $\mathbf{B}\Delta$  for which  $\mathbf{B} \rightarrow \mathbf{B}\Delta$  is a braided strict monoidal functor and  $\Delta \rightarrow \mathbf{B}\Delta$  is strict monoidal. The symmetric strict monoidal category  $\mathbf{P}\Delta$  is defined similarly; in fact,  $\mathbf{P}\Delta$  is equivalent to the category whose objects are finite sets and whose morphisms are functions with the structure of linear order on each fibre<sup>3</sup>.

Recall from [7] that lax monoids in a Gray monoid  $\mathcal{M}$  amount to strict-monoidal lax functors  $M: \Delta \rightarrow \mathcal{M}$ . A *braided* lax monoid in a braided (see [6]) Gray monoid  $\mathcal{M}$  is a braided strict-monoidal lax functor  $M: \mathbf{B}\Delta \rightarrow \mathcal{M}$  whose composition constraints

$$\mu_{\theta, \phi}: M(\phi)M(\theta) \rightarrow M(\phi, \theta)$$

are identities whenever  $\theta$  is invertible. In particular, this last condition implies that the composite of  $M$  with  $\mathbf{B} \rightarrow \mathbf{B}\Delta$  is a 2-functor. Of course the composite of  $M$  with the strict-monoidal functor  $\Delta \rightarrow \mathbf{B}\Delta$  is the *underlying lax monoid* of  $M$ ; we say that  $M$  provides a *braiding* for the underlying lax monoid.

<sup>2</sup> This misprint was pointed out to us by Teimuraz Pirashvili.

<sup>3</sup> In the late 1980s, André Joyal pointed out this description of  $\mathbf{P}\Delta$  to the second author. He also remarked that there is a corresponding fact for  $\mathbf{B}\Delta$  whereby “linear order” is replaced by “plane configuration”.

Similarly, a *symmetric* lax monoid in a sylleptic (see [6]) Gray monoid  $\mathcal{M}$  is a sylleptic strict-monoidal lax functor  $M : \mathbf{P}\Delta \rightarrow \mathcal{M}$  whose composition constraints

$$\mu_{\theta, \phi} : M(\phi)M(\theta) \rightarrow M(\phi, \theta)$$

are identities whenever  $\theta$  is invertible.

A *braided* [respectively, *symmetric*]  $\mathcal{V}$ -substitute is a braided [respectively, symmetric] lax monoid in  $(\mathcal{V}\text{-}\mathbf{Mod})^{\text{op}}$ .

More explicitly, a braiding for a substitute  $\mathcal{A}$  assigns to each natural number  $n$  and each braid  $\alpha$  on  $n$  strings, a  $\mathcal{V}$ -natural family of isomorphisms

$$\gamma_\alpha : P_n(A_1, \dots, A_n; X) \xrightarrow[\sim]{\quad} P_n(A_{\alpha(1)}, \dots, A_{\alpha(n)}; X),$$

such that  $\gamma_{\alpha'} \circ \gamma_\alpha = \gamma_{\alpha'\alpha}$  and the following two diagrams commute:

$$\begin{array}{ccc} P_n(A_{\alpha(\bullet)}; A) \otimes \bigotimes_{i=1}^n P_{m_{\alpha(i)}}(A_{\alpha(i)\bullet}; A_{\alpha(i)}) & \xrightarrow{\mu_{\gamma_\alpha}} & P_{\gamma_\alpha}(A_{\alpha(\bullet)\bullet}; A) \\ \uparrow \gamma_\alpha \otimes c_\alpha^{-1} & & \uparrow \gamma_{\alpha^\sharp} \\ P_n(A_\bullet; A) \otimes \bigotimes_{i=1}^n P_{m_i}(A_{i\bullet}; A_i) & \xrightarrow{\mu_\sharp} & P_\sharp(A_{\bullet\bullet}; A) \\ & & \downarrow \gamma_{x_1 + \dots + x_n} \\ P_n(A_\bullet; A) \otimes \bigotimes_{i=1}^n P_{\alpha_i(m_i)}(A_{i\alpha_i(\bullet)}; A_i) & \xrightarrow{\mu_\sharp} & P_m(A_{1\alpha_1(\bullet)}, \dots, A_{n\alpha_n(\bullet)}) \end{array}$$

The braiding is a *symmetry* when  $\gamma_\alpha \gamma_\alpha = 1$  if  $\alpha$  is the basic left-over-right cross-over of two strings.

A  $\mathcal{V}$ -operad  $T$  is *braided* [*symmetric*] when the  $\mathcal{V}$ -substitute  $\Sigma T$  is equipped with a braiding [symmetry]. Symmetric operads are the original operads of May [16]. Braided operads have been used for example by Fiedorowicz [9] and Tamarkin [18].

**Proposition 4.1.** *Suppose  $\mathcal{A}$  is a small braided [symmetric]  $\mathcal{V}$ -substitute and  $\mathcal{X}$  is a cocomplete braided [symmetric] lax monoidal  $\mathcal{V}$ -category. Then the standard convolution  $[\mathcal{A}, \mathcal{X}]$  has a canonical braiding [symmetry] induced on coends by the following isomorphisms:*

$$\begin{aligned} & P_n(A_1, \dots, A_n; -) \otimes \bigotimes_n (F_1 A_1, \dots, F_n A_n) \\ & \xrightarrow{\gamma_\alpha \otimes c_\alpha} P_n(A_{\alpha(1)}, \dots, A_{\alpha(n)}; -) \otimes \bigotimes_n (F_{\alpha(1)} A_{\alpha(1)}, \dots, F_{\alpha(n)} A_{\alpha(n)}). \end{aligned}$$

**Proposition 4.2.** *Suppose  $\mathcal{A}$  is a small braided [symmetric]  $\mathcal{V}$ -substitute. Then the non-standard convolution  $[\mathcal{A}^{\text{op}}, \mathcal{V}]$  has a canonical braiding [symmetry] induced on*

ends by the following isomorphisms:

$$\left[ F_1 A_1 \otimes \cdots \otimes F_n A_n, \int^A P_n(A_1, \dots, A_n; A) \otimes FA \right] \\ \xrightarrow{[c_z, \int \gamma_z \otimes 1]} \left[ F_{z(1)} A_{z(1)} \otimes \cdots \otimes F_{z(n)} A_{z(n)}, \int^A P_n(A_{z(1)}, \dots, A_{z(n)}; A) \otimes FA \right].$$

We conclude this section with a construction in the case where  $\mathcal{V}$  is Cartesian closed.

Each substitute  $\mathcal{A}$  can be *symmetrized*. Explicitly, we define a new substitute structure  $\mathcal{Q} = \text{sym}(\mathcal{P})$  on the same underlying category  $\mathcal{A}$  by

$$\mathcal{Q}_n(A_1, \dots, A_n; A) = \sum_{\sigma \in P_n} P_n(A_{\sigma(1)}, \dots, A_{\sigma(n)}; A).$$

To define the substitution operation we define its composite with the coprojection from

$$P_n(A_{\sigma(1)}, \dots, A_{\sigma(n)}; A) \times \prod_{r=0}^n P_{m_r}(A_{r\sigma_r(1)}, \dots, A_{r\sigma_r(m_r)}; A_r)$$

to

$$\mathcal{Q}_n(A_1, \dots, A_n; A) \times \prod_{r=0}^n \mathcal{Q}_{m_r}(A_{r1}, \dots, A_{rm_r}; A_r) \quad \text{for } \sigma \in \mathbf{P}_n \text{ and } \sigma_r \in \mathbf{P}_{m_r},$$

to be the composite of the canonical isomorphism  $1 \times c_{\sigma^{-1}}$  into

$$P_n(A_{\sigma(1)}, \dots, A_{\sigma(n)}; A) \times \prod_{r=0}^n P_{m_{\sigma(r)}}(A_{\sigma(r)\sigma_{\sigma(r)}(1)}, \dots, A_{\sigma(r)\sigma_{\sigma(r)}(m_{\sigma(r)})}; A_{\sigma(r)}),$$

the substitution  $\mu_{\xi_\sigma}^\varepsilon$ , and the coprojection into  $\mathcal{Q}_m(A_{11}, \dots, A_{nm_n}; A)$  for the permutation  $\sigma_{m_{\sigma(1)}} + \cdots + \sigma_{m_{\sigma(n)}}$  of  $m_{\sigma(1)} + \cdots + m_{\sigma(n)} = m$ . Noting that

$$\mathcal{Q}_1(A_1; A) = P_1(A_1; A),$$

we take the same unit as before. The three substitute axioms can be verified. Moreover, we have a natural isomorphism

$$\gamma_\tau : \mathcal{Q}_n(A_1, \dots, A_n; A) \xrightarrow[\sim]{} \mathcal{Q}_n(A_{\tau(1)}, \dots, A_{\tau(n)}; A)$$

whose composite with the coprojection from  $P_n(A_{\sigma(1)}, \dots, A_{\sigma(n)}; A)$  for  $\sigma \in \mathbf{P}_n$  is the coprojection for  $\tau\sigma \in \mathbf{P}_n$ . The three axioms for a symmetry can be verified.

## 5. Braided substitute morphisms

Suppose  $\mathcal{A}$  and  $\mathcal{X}$  are braided  $\mathcal{V}$ -substitutes. A  $\mathcal{V}$ -substitute morphism  $F: \mathcal{A} \rightarrow \mathcal{X}$  is called *braided* when the following square commutes for all objects  $A_1, \dots, A_n$  and  $A$  of  $\mathcal{A}$ , and all braids  $\alpha$

$$\begin{array}{ccc} P_n(A_1, \dots, A_n; A) & \xrightarrow{\gamma_\alpha} & P_n(A_{\alpha(1)}, \dots, A_{\alpha(n)}; A) \\ \phi_n \downarrow & & \downarrow \phi_n \\ P_n(FA_1, \dots, FA_n; FA) & \xrightarrow{\gamma_\alpha} & P_n(FA_{\alpha(1)}, \dots, FA_{\alpha(n)}; FA) \end{array}$$

If  $T$  is a braided [symmetric]  $\mathcal{V}$ -operad then a *T-algebra* in  $\mathcal{A}$  is defined to be a braided [symmetric]  $\mathcal{V}$ -substitute morphism  $A: \Sigma T \rightarrow \mathcal{A}$ . For example, when  $T$  is the symmetric operad  $I$ , an  $I$ -algebra in  $\mathcal{A}$  is called a *commutative monoid* in  $\mathcal{A}$ . The commutative monoids in the standard convolution  $[\mathcal{A}, \mathcal{X}]$  are braided substitute morphisms.

In the case where  $\mathcal{V}$  is the category of complex vector spaces, there is a symmetric  $\mathcal{V}$ -operad **lie** whose algebras in  $\mathcal{V}$  are complex Lie algebras. We define a *Lie algebra* in a  $\mathcal{V}$ -substitute  $\mathcal{A}$  to be a **lie**-algebra in  $\mathcal{A}$ . Lie algebras in the non-standard convolution structure  $[H^{\text{op}}, \mathcal{V}]$  for a symmetric finite-dimensional Hopf algebra  $H$  were extensively studied in [1]. Another connection between **lie** and substitutes will be addressed in [2].

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