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An Example Paper

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Abstract

This is a short example to show the basics of using the ENTCS style macro files. Ample examples of how files should look may be found among the published volumes of the series at the ENTCS Home Page `http://www.elsevier.com/locate/entcs`.

Keywords: Please list keywords from your paper here, separated by commas.

Test important : il faut qu'on obtienne les bons concepts lorsqu'on se restreint à des sesquicatégories qui sont des 2-catégories

1 General definitions on sesquicategories

Definition 1.1 A *sesquicategory* is given by a category $\mathcal{C}$ together with a functor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Cat}$ such that the composite $ob \circ H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ with the underlying-set functor is equal to the hom functor of $\mathcal{C}$.

Definition 1.2 A *sesquicategory* $\mathcal{S}$ is given by

(i) a set of 0-cells S_0,

(ii) a set of 1-cells S_1 together with source and target applications $s_1, t_1 : S_1 \to S_0$,

(iii) a set of 2-cells S_2 together with source and target applications $s_2, t_2 : S_2 \to S_1$,

(iv) a composition law

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$$\begin{aligned} \circ : S_1 \times S_1 &\to S_1 \\ (f,g) &\mapsto g \circ f \end{aligned}$$

whenever $t_1 f = s_1 g$, and in that case $s_1(g \circ f) = s_1 f$ and $t_1(g \circ f) = t_1 g$,

(v) a composition law

$$\begin{aligned} \bullet : S_2 \times S_2 &\to S_2 \\ (\alpha,\beta) &\mapsto \beta \bullet \alpha \end{aligned}$$

whenever $t_2\alpha = s_2\beta$, and in that case $s_2(\beta \bullet \alpha) = s_2\alpha$ and $t_2(\beta \circ \alpha) = t_2\beta$ and

(vi) an action on the left

$$\begin{aligned} S_2 \times S_1 &\to S_2 \\ (\alpha,h) &\mapsto h.\alpha \end{aligned}$$

whenever $t_1 \circ s_2\alpha = s_1 h$, and in that case $s_2(h.\alpha) = h \circ (s_2\alpha)$ and $t_2(h.\alpha) = h \circ (t_2\alpha)$,

(vii) an action on the right

$$\begin{aligned} S_2 \times S_1 &\to S_2 \\ (\alpha,h) &\mapsto \alpha.h \end{aligned}$$

whenever $t_1 h = s_1 \circ s_2\alpha$, and in that case $s_2(\alpha.h) = (s_2\alpha) \circ h$ and $t_2(\alpha.h) = (t_2\alpha) \circ h$,

satisfying the following assumptions :

(i) The source and target applications satisfy the globular relations :

$$s_1 \circ s_2 = s_1 \circ t_2 \qquad t_1 \circ s_2 = t_1 \circ t_2.$$

(ii) The composition law $\circ$ is associative and for all 0-cell x, there is a 1-cell id_x such that for all 1-cells f and g such that $s_1 f = x$ and $t_1 g = x$, $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_x \circ g = g$.

(iii) The composition law $\bullet$ is associative and for all 1-cell f, there is a 2-cell Id_f such that for all 2-cells α and β such that $s_2\alpha = f$ and $t_2\beta = f$, $\alpha \bullet \mathrm{Id}_f = \alpha$ and $\mathrm{Id}_f \bullet \beta = \beta$.

(iv) For all 1-cells h_1 and h_2 and 2-cells α and β such that $t_1 h_1 = s_1 \circ s_2\alpha$, $t_2\alpha = s_2\beta$ and $t_1 \circ t_2\beta = s_1 h_2$,

$$(h_2.\beta) \bullet (h_2.\alpha) = h_2.(\beta \bullet \alpha) \qquad (\beta.h_1) \bullet (\alpha.h_1) = (\beta \bullet \alpha).h_1$$

(v) For all 1-cells f, h_1 and h_2 such that $t_1 h_1 = s_1 f$ and $t_1 f = s_1 h_2$,

$$(h_2.\mathrm{Id}_f) = \mathrm{Id}_{h_2 \circ f} \qquad (\mathrm{Id}_f .h_1) = \mathrm{Id}_{f \circ h_1}$$

(vi) For all 1-cells f_1, f_2, g_1, g_2 and 2-cell α such that $t_1 g_2 = s_1 g_1$, $t_1 g_1 = s_1 \circ s_2\alpha$, $t_1 \circ s_2\alpha = s_1 f_1$ and $t_1 f_1 = s_1 f_2$,

$$f_2.(f_1.\alpha) = (f_2 \circ f_1).\alpha \qquad (\alpha.g_1).g_2 = \alpha.(g_1 \circ g_2) \qquad (f_1.\alpha).g_1 = f_1.(\alpha.g_1)$$

(vii) For all 0-cells x and y and 2-cell α such that $s_1 \circ s_2\alpha = x$ and $t_1 \circ s_2\alpha = y$,

$$\mathrm{id}_y .\alpha = \alpha \qquad \alpha.\mathrm{id}_x = \alpha$$

Remark 1.3 These relations justify that we denote $(h_2.\alpha).h_1$ by $h_2.\alpha.h_1$. We often write $f : x \to y$ for f in S_1 such that $s_1 f = x$ and $t_1 f = y$ and $\alpha : f \Rightarrow g$ for α in S_2 such that $s_2\alpha = f$ and $t_2\alpha = g$. Besides, we may also combine notations and write $\alpha : f \Rightarrow g : x \to y$.

Lemma 1.4 *The two definitions of sesquicategory are equivalent.*

Proof. Given a sesquicategory $\mathcal{S}$ according to the second definition, we define the category $\mathcal{C}$ as the category with set of objects S_0, set of morphisms S_1, composition law $\circ$ and identities id_x for all objects x in S_0 and the functor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Cat}$ as follows. For any object (x, y) of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, $H(x, y)$ is defined as the category with set of objects

$$S_1(x, y) = \{f : x \to y \in S_1\}$$

and set of morphisms

$$S_2(x, y) = \{\alpha : f \Rightarrow g : x \to y \in S_2\}$$

and where the composition is $\bullet$ with identities Id_f for f in $S_1(x, y)$. For any morphism $(f^{\mathrm{op}} : x' \to x, g : y' \to y)$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we define $H(f^{\mathrm{op}}, g)$ as the functor

$$\begin{aligned} F_{f,g} : \qquad H(x', y') &\to H(x, y) \\ h &\mapsto g \circ h \circ f \\ (\alpha : h_1 \Rightarrow h_2) &\mapsto (g.\alpha.f : g \circ h_1 \circ f \Rightarrow g \circ h_2 \circ f) \end{aligned}$$

where $f : x \to x'$ is the morphism of $\mathcal{C}$ corresponding to f^{op} in $\mathcal{C}^{\mathrm{op}}$. This is indeed a functor : let us consider β and α in $S_2(x', y')$ such that $s_2(\beta) = t_2(\alpha)$, then

$$\begin{aligned} F_{f,g}(\beta \bullet \alpha) &= g.(\beta \bullet \alpha).f \\ &= (g.\beta.f) \bullet (g.\alpha.f) \\ &= F_{f,g}(\beta) \bullet F_{f,g}(\alpha) \end{aligned}$$

Let us consider $h : x' \to y'$ in S_1, then

$$\begin{aligned} F_{f,g}(\mathrm{Id}_h) &= g.\,\mathrm{Id}_h\,.f \\ &= \mathrm{Id}_{g \circ h \circ f} \\ &= \mathrm{Id}_{F_{f,g} h} \end{aligned}$$

There remains to show that $ob \circ H = hom$. By definition $ob \circ H(x, y) = S_1(x, y) = hom_{\mathcal{C}}(x, y)$. Considering $(f^{\mathrm{op}} : x' \to x, g : y' \to y)$ a morphism in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we get that $ob \circ H(f^{\mathrm{op}}, g) = ob(F_{f,g})$ is the function between the underlying sets induced by the functor $F_{f,g}$ which is $hom_{\mathcal{C}}(f, g)$.

Given a sesquicategory $\mathcal{C}$ equipped with a functor $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Cat}$ according to the first definition, we define $\mathcal{S}$ as

(i) the set of 0-cells S_0 is the set of objects of $\mathcal{C}$,

(ii) the set of 1-cells S_1 is the set of morphisms of $\mathcal{C}$,

(iii) the set of 2-cells S_2 is defined as the union of all sets $S_2(x, y)$ for x, y in S_0 and where $S_2(x, y)$ is defined as the set of morphisms of the category $H(x, y)$,

(iv) the composition law $\circ$ and the corresponding identities id_x for x in S_0 are given by the composition and identities in the category $\mathcal{C}$,

(v) the composition $\bullet$ for 2-cells in $S_2(x,y)$ and the corresponding identities Id_f for f in $S_1(x,y)$ are given by the composition and identities in the category $H(x,y)$,

(vi) for g in $S_1(y,z)$ and α in $S_2(x,y)$, $H(\mathrm{id}_x, g)$ is a functor $H(x,y) \to H(x,z)$ and the left action is given by

$$g.\alpha = H(\mathrm{id}_x, g)\alpha$$

(vii) Similarly, for f in $S_1(x,y)$ and α in $S_2(y,z)$, the right action is given by

$$\alpha.f = H(f, \mathrm{id}_z)\alpha$$

□

Lemma 1.5 *Any 2-cell of a sesquicategory can be written as*

$$\alpha_1 \bullet \dots \bullet \alpha_k$$

where $\alpha_i = f_i.\beta_i.g_i$ *where* f_i *and* g_i *are in* P_1^* *and* β_i *is in* P_2*.*

Definition 1.6 A *strict functor of sesquicategory* $F : \mathcal{S} \to \mathcal{S}'$ is given by three applications $F_i : \mathcal{S}_i \to \mathcal{S}'_i$ $(i = 0,1,2)$ such that for all x in $\mathcal{S}_0$, f, g in $\mathcal{S}_1$ and α, β in $\mathcal{S}_2$,

$$\begin{array}{rlrl}
s'_1(Ff) &= F(s_1 f) & t'_1(Ff) &= F(t_1 f) \\
s'_2(F\alpha) &= F(s_2\alpha) & t'_2(F\alpha) &= F(t_2\alpha) \\
F(g) \circ F(f) &= F(g \circ f) & F\,\mathrm{id}_x &= \mathrm{id}_{Fx} \\
F(\alpha) \bullet F(\beta) &= F(\alpha \bullet \beta) & F\,\mathrm{Id}_f &= \mathrm{Id}_{Ff} \\
F(f).F(\alpha).F(g) &= F(g.\alpha.f) & &
\end{array}$$

Definition 1.7 A *weak functor of sesquicategory* $F : \mathcal{S} \to \mathcal{S}'$ is given by three applications $F_i : \mathcal{S}_i \to \mathcal{S}'_i$ $(i = 0,1,2)$ such that for all x in $\mathcal{S}_0$, f, g in $\mathcal{S}_1$ and α, β in $\mathcal{S}_2$,

$$\begin{array}{rlrl}
s'_1(Ff) &= F(s_1 f) & t'_1(Ff) &= F(t_1 f) \\
s'_2(F\alpha) &= F(s_2\alpha) & t'_2(F\alpha) &= F(t_2\alpha) \\
F(\alpha) \bullet F(\beta) &= F(\alpha \bullet \beta) & F\,\mathrm{Id}_f &= \mathrm{Id}_{Ff}
\end{array}$$

Such that there exist isomorphisms :

$$\phi_{f,g} : F(g) \circ F(f) \to F(g \circ f) \qquad \phi_x : F\,\mathrm{id}_x \to \mathrm{id}_{Fx}$$

making the following diagrams commute for all $\beta : g \Rightarrow g'$ and $\alpha : f \Rightarrow f'$:

$$\begin{array}{ccc}
F(g) \circ F(f) & \xrightarrow{\phi_{f,g}} & F(g \circ f) \\
{\scriptstyle F\beta.Ff)}\downarrow & & \downarrow{\scriptstyle F(\beta.f)} \\
F(g') \circ F(f) & \xrightarrow{\phi_{f,g'}} & F(g' \circ f)
\end{array}
\qquad
\begin{array}{ccc}
F(g) \circ F(f) & \xrightarrow{\phi_{f,g}} & F(g \circ f) \\
{\scriptstyle Fg.F\alpha}\downarrow & & \downarrow{\scriptstyle F(g.\alpha)} \\
F(g) \circ F(f') & \xrightarrow{\phi_{f',g}} & F(g \circ f')
\end{array}$$

$$F(\mathrm{id}_x) \circ Ff \underset{\phi_x.Ff}{\overset{\phi_{f,\mathrm{id}_x}}{\rightrightarrows}} Ff \qquad Fg \circ F(\mathrm{id}_x) \underset{Fg.\phi_x}{\overset{\phi_{\mathrm{id}_x,g}}{\rightrightarrows}} Fg$$

and for all h, g, f in S_1 :

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \xrightarrow{Fh.\phi_{f,g}} & Fh \circ F(g \circ f) \\ {\scriptstyle \phi_{g,h}.Ff}\downarrow & & \downarrow{\scriptstyle \phi_{g\circ f,h}} \\ F(h \circ g) \circ Ff & \xrightarrow[\phi_{f,h\circ g}]{} & F(h \circ g \circ f) \end{array}$$

Definition 1.8 *equivalence of sesquicategories*

Definition 1.9 A weak functor of sesquicategory $F : \mathcal{S} \to \mathcal{S}'$ is an *equivalence of sesquicategories* if and only if it is

- essentially surjective on 0-cells : for all y in $\mathcal{S}'_0$, there exists x in $\mathcal{S}_0$ such that Fx and y are isomorphic,
- essentially full on 1-cells : for all x and y in $\mathcal{S}_0$, for all g in $\mathcal{S}'_1(Fx, Fy)$, there exists f in $\mathcal{S}_1(x, y)$ such that Ff and g are isomorphic,
- and fully faithful on 2-cells : for all x and y in $\mathcal{S}_0$, the functor induced by F between the categories $\mathcal{S}(x, y)$ and $\mathcal{S}'(Fx, Fy)$ is full and faithful.

Lemma 1.10 *The two definitions of equivalence of sesquicategories are equivalent.*

Proof. TODO □

2 Presentation of sesquicategories

Definition 2.1 A *presentation of a sesquicategory* is defined as the 4-uple of sets (P_0, P_1, P_2, P_3) such that (P_0, P_1, P_2) is a presentation of category. It generates a sesquicategory $\mathcal{S}'$ where $S'_0 = P_0$, S'_1 is the set of morphisms of the category generated by the presentation of category (P_0, P_1, P_2) (it is often denoted by P_1^*) and S'_2 is the set of 2-cells generated by P_2 and closed under composition $\bullet$ and context .. The set P_3 is a set of relations between elements of S'_2 such that for any relation (α, β) in P_3, then $s_2\alpha = s_2\beta$ and $t_2\alpha = t_2\beta$. The set of relations in P_3 generates a congruence $\equiv$ as the smallest congruence closed under composition $\bullet$ and context . such that $\alpha \equiv \beta$ for (α, β) in P_3

The *sesquicategory presented* is defined as the sesquicategory with set of 0-cells S'_0, of 1-cells S'_1 and 2-cells $S'_2/\equiv$ and compositions $\circ$, $\bullet$ and context . and it is denoted $\|P\|$.

FC: quelques détails à vérifier quand même...

Definition 2.2 A *2-category* $\mathcal{C}$ consists of

(i) a set $\mathcal{C}_0$ of 0-*cells*,

(ii) a set $\mathcal{C}_1$ of 1-*cells* and source and target functions $s_1, t_1 : \mathcal{C}_1 \to \mathcal{C}_0$,

(iii) a set $\mathcal{C}_2$ of 2-*cells* together with source and target functions $s_2, t_2 : \mathcal{C}_2 \to \mathcal{C}_1$,

(iv) *horizontal identities* $\mathrm{id}_x : x \to x \in \mathcal{C}_1$ indexed by $x \in \mathcal{C}_0$,

(v) *vertical identities* $\mathrm{Id}_f : f \Rightarrow f \in \mathcal{C}_2$ indexed by $f \in \mathcal{C}_1$,

(vi) *horizontal composition* function $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$

(vii) *horizontal composition* function $\circ : \mathcal{C}_2 \times_{\mathcal{C}_0} \mathcal{C}_2 \to \mathcal{C}_2$

(viii) *vertical composition* functions $\bullet : \mathcal{C}_2 \times_{\mathcal{C}_1} \mathcal{C}_2 \to \mathcal{C}_2$,

such that

(i) globular axioms hold: $s_1 \circ s_2 = s_1 \circ t_2$ and $t_1 \circ s_2 = t_1 \circ t_2$,

(ii) identities are neutral elements: $\mathrm{id}_y \circ f = f = f \circ \mathrm{id}_x$,

(iii) identities are neutral elements: $\mathrm{Id}_g \bullet \alpha = \alpha = \alpha \bullet \mathrm{Id}_f$,

(iv) horizontal composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$,

(v) horizontal composition is associative: $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$,

(vi) vertical composition is associative: $\gamma \bullet (\beta \bullet \alpha) = (\gamma \bullet \beta) \bullet \alpha$,

(vii) identities are compatible with composition: $\mathrm{id}_{g \circ f} = \mathrm{id}_g \circ \mathrm{id}_f$,

(viii) the exchange law holds: $(\beta' \bullet \beta) \circ (\alpha' \bullet \alpha) = (\beta' \circ \alpha') \bullet (\beta \circ \alpha)$

Lemma 2.3 *Given a presentation (P_0, P_1, P_2, P_3) such that for all $\alpha : f_1 \Rightarrow f_2 : x \to y$ and $\beta : h_1 \Rightarrow h_2 : z \to w$ in P_2 and $g : y \to z$ in P_1^*, there is a relation in P_3*

$$(\beta.g.f_2) \bullet (h_1.g.\alpha) \equiv (h_2.g.\alpha) \bullet (\beta.g.f_1),$$

then the sesquicategory presented is a 2-category.

Proof. The horizontal composition on 1-cells is $\circ$. The vertical composition on 2-cells is $\bullet$. The horizontal composition between two 2-cells $\alpha : f_1 \Rightarrow f_2$ and $\beta : g_1 \Rightarrow g_2$ with $s_1 g_1 = t_1 f_1$ is defined as

$$\beta \circ \alpha = (g_2.\alpha) \bullet (\beta.f_1)$$

This horizontal composition is associative : given three 2-cells $\alpha : f_1 \Rightarrow f_2$, $\beta : g_1 \Rightarrow g_2$ and $\gamma : h_1 \Rightarrow h_2$ with $s_1 g_1 = t_1 f_1$ and $s_1 h_1 = t_1 g_1$,

$$\begin{aligned}
\gamma \circ (\beta \circ \alpha) &= (h_2.(\beta \circ \alpha)) \bullet (\gamma.(g_1 \circ f_1)) \\
&= (h_2.((g_2.\alpha) \bullet (\beta.f_1))) \bullet (\gamma.(g_1 \circ f_1)) \\
&= ((h_2.g_2.\alpha) \bullet (h_2.\beta.f_1)) \bullet (\gamma.g_1.f_1) \\
&= (h_2.g_2.\alpha) \bullet ((h_2.\beta.f_1) \bullet (\gamma.g_1.f_1)) \\
&= (h_2.g_2.\alpha) \bullet (((h_2.\beta) \bullet (\gamma.g_1)).f_1) \\
&= ((h_2 \circ g_2).\alpha) \bullet (\gamma \circ \beta).f_1) \\
&= (\gamma \circ \beta) \circ \alpha
\end{aligned}$$

The identities also verify the relation for horizontal composition

$$\begin{aligned}
\mathrm{Id}_g \circ \mathrm{Id}_f &= (g.\,\mathrm{Id}_f) \bullet (\mathrm{Id}_g\,.f) \\
&= \mathrm{Id}_{g \circ f} \bullet \mathrm{Id}_{g \circ f} \\
&= \mathrm{Id}_{g \circ f}
\end{aligned}$$

To prove the exchange law, we prove that under the hypothesis of the theorem, we prove that for any $\alpha : f_1 \Rightarrow f_3$ and $\beta : g_1 \to g_3$ in P_2^*, then

$$(\beta.f_2) \bullet (g_1.\alpha) = (g_3.\alpha) \bullet (\beta.f_1)$$

This would prove that

$$(\beta \circ \mathrm{Id}_{f_3}) \bullet (\mathrm{Id}_{g_1} \circ \alpha) = (\mathrm{Id}_{g_3} \circ \alpha) \bullet (\beta \circ \mathrm{Id}_{f_1})$$

which is called the Godement law and which is equivalent to the exchange law.

This is proven by induction on the structure of the 2-cell, using Lemma 1.5 and the associativity of $\bullet$.

For $\alpha = \alpha_2 \bullet \alpha_1$ with $\alpha_2 : f_2 \Rightarrow f_3$ and $\alpha_1 : f_1 \Rightarrow f_2$ in P_2^* (and similarly $\beta = \beta_2 \bullet \beta_1$),

$$\begin{aligned}
\big((\beta_2 \bullet \beta_1).f_3\big) \bullet \big(g_1.(\alpha_2 \bullet \alpha_1)\big) &= (\beta_2.f_3) \bullet \big((\beta_1.f_3) \bullet (g_1.\alpha_2)\big) \bullet (g_1.\alpha_1) \\
&= \big((\beta_2.f_3) \bullet (g_2.\alpha_2)\big) \bullet \big((\beta_1.f_2) \bullet (g_1.\alpha_1)\big) \\
&= (g_3.\alpha_2) \bullet \big((\beta_2.f_2) \bullet (g_2.\alpha_1)\big) \bullet (\beta_1.f_1) \\
&= \big((g_3.\alpha_2) \bullet (g_3.\alpha_1)\big) \bullet \big((\beta_2.f_1) \bullet (\beta_1.f_1)\big) \\
&= \big(g_3.(\alpha_2 \bullet \alpha_1)\big) \bullet \big((\beta_2 \bullet \beta_1).f_1\big)
\end{aligned}$$

For $\alpha = h_2.\alpha'.h_1$ and $\beta = k_2.\beta'.k_1$, with $f_i = h_2 \circ f_i' \circ h_1$ and $g_i = k_2 \circ g_i' \circ k_1$ (with $i = 1, 3$),

$$\begin{aligned}
(\beta.f_3) \bullet (g_1.\alpha) &= \big(k_2.\beta'.(k_1 \circ f_3)\big) \bullet \big((g_1 \circ h_2).\alpha'.h_1\big) \\
&= \big(k_2.(\beta'.(k_1 \circ h_2 \circ f_3')).h_1\big) \bullet \big(k_2.((g_1' \circ k_1 \circ h_2).\alpha').h_1\big) \\
&= k_2.\big((\beta'.(k_1 \circ h_2 \circ f_3')) \bullet ((g_1' \circ k_1 \circ h_2).\alpha')\big).h_1 \\
&= k_2.\big(((g_3' \circ k_1 \circ h_2).\alpha') \bullet (\beta'.(k_1 \circ h_2 \circ f_1'))\big).h_1 \\
&= (g_3.\alpha) \bullet (\beta.f_1)
\end{aligned}$$

□

Definition 2.4 A *presentation modulo of sesquicategory* is a presentation of sesquicategory $P = (P_0, P_1, P_2, P_3)$ together with a set $\tilde{P}_2 \subset P_2$. It is denoted $(P, \tilde{P}_2)$.

Definition 2.5 hom-cat of a sesquicat ? hom-functor ?

Lemma 2.6 *The hom-category $\mathcal{S}(x, y)$ of a Sesquicategory $\mathcal{S}$ presented by (P_0, P_1, P_2, P_3) is presented by (Q_1, Q_2, Q_3) where*

$$\begin{aligned}
&Q_1 = P_1^*(x, y) = \{f \mid f : x \to y \in P_1^*\} \\
&Q_2 = P_2^{CC}(x, y) = \{h_1.\alpha.h_2 : f \Rightarrow g : x \to y \mid h_1, h_2 \in P_1^*, \alpha \in P_2\} \\
&Q_3 = P_3^{CC}(x, y) = \{h_1.\alpha.h_2 \Rrightarrow h_1.\beta.h_2 : f \Rightarrow g : x \to y \mid h_1, h_2 \in P_1^*, \alpha, \beta \in P_2^*\}
\end{aligned}$$

Proof. TODO □

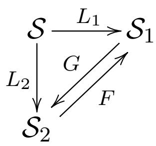
3 Localization and quotient of sesquicategories

Definition 3.1 **Definition 3.2** The *localization* of a sesquicategory $\mathcal{S}$ by a set Σ of 2-cells of $\mathcal{S}$ is a sesquicategory $\mathcal{S}[\Sigma^{-1}]$ together with a weak *quotient functor* of sesquicategory $Q : \mathcal{S} \to \mathcal{S}[\Sigma^{-1}]$ sending the elements of Σ to isomorphisms of $\mathcal{S}'$, such that for every weak functor of sesquicategory $F : \mathcal{S} \to \mathcal{S}'$ sending the elements of Σ to isomorphisms, there exists a unique weak functor of sesquicategory $\tilde{F}$ such that $\tilde{F} \circ Q = F$.

Definition 3.3 An *isomorphism of sesquicategories* $\mathcal{S}$ and $\mathcal{S}'$ is given by two weak functors $F : \mathcal{S} \to \mathcal{S}'$ and $G : \mathcal{S}' \to \mathcal{S}$ such that $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

Lemma 3.4 *Given three sesquicategories $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$, such that $\mathcal{S}_1$ is a localization of the sesquicategory $\mathcal{S}$ by a set Σ of 2-cells of $\mathcal{S}$. The sesquicategory $\mathcal{S}_2$ is a localization of $\mathcal{S}$ by Σ if and only if $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic.*

Proof. Let us assume that $\mathcal{S}_2$ is also a localization of $\mathcal{S}$ by Σ, then there are two localization weak functors $L_i : \mathcal{S} \to \mathcal{S}_i$ $(i = 1, 2)$ sending the 2-cells in Σ to isomorphisms in $\mathcal{S}_i$. By universal property, there exist two weak functors $F : \mathcal{S}_2 \to \mathcal{S}_1$ and $G : \mathcal{S}_1 \to \mathcal{S}_2$ such that $F \circ L_2 = L_1$ and $G \circ L_1 = L_2$



By composing the two equalities, we get $G \circ F \circ L_2 = L_2$. Using the fact that $\mathcal{S}_2$ is a localization of $\mathcal{S}$, we get that there is a unique weak functor $\mathrm{id} : \mathcal{S}_2 \to \mathcal{S}_2$ such that $\mathrm{id} \circ L_2 = L_2$. This means that $G \circ F = \mathrm{id}_{\mathcal{S}_2}$. Similarly, we get that $F \circ G = \mathrm{id}_{\mathcal{S}_1}$. Therefore, there is an isomorphism of sesquicategory between $\mathcal{S}_1$ and $\mathcal{S}_2$.

Conversely, let us assume that there are two weak functors of sesquicategory $F : \mathcal{S}_2 \to \mathcal{S}_1$ and $G : \mathcal{S}_1 \to \mathcal{S}_2$ such that $G \circ F = \mathrm{id}_{\mathcal{S}_2}$ and $F \circ G = \mathrm{id}_{\mathcal{S}_1}$. We define the weak functor $L_2 : \mathcal{S} \xrightarrow{L_1} \mathcal{S}_1 \xrightarrow{G} \mathcal{S}_2$. By construction, the elements in Σ are sent to isomorphisms in $\mathcal{S}_2$ (this relies on the fact that $L_2(\alpha \bullet \beta) = L_2\alpha \bullet L_2\beta$).
To show the universal property, let us consider a weak functor $H : \mathcal{S} \to \mathcal{S}'$. By universal property of $\mathcal{S}_1$, there exists a weak functor $H_1 : \mathcal{S}_1 \to \mathcal{S}'$ such that $H_1 \circ L_1 = H$. We set $H_2 : \mathcal{S}_2 \xrightarrow{F} \mathcal{S}_1 \xrightarrow{H_1} \mathcal{S}'$. We get

$$H_2 \circ L_2 = H_1 \circ F \circ G \circ L_1 = H_1 \circ L_1 = H.$$

This weak functor is unique : assuming that both $H, 2, H_3 : \mathcal{S}_2 \to \mathcal{S}'$ satisfy the universal property, then $H_2 \circ F$ and $H_3 \circ F$ satisfy the universal property of the localization for $\mathcal{S}_1$, which means that $H_2 \circ F = H_3 \circ F$ and by composing with G and using $F \circ G = \mathrm{id}$, we get $H_2 = H_3$. □

Lemma 3.5 *Given a sesquicategory $\mathcal{S}$ presented by (P_0, P_1, P_2, P_3) and a subset $\tilde{P}_2$ of P_2, then the localization $\mathcal{S}[\tilde{P}_2^{-1}]$ is presented by the presentation $(P_0, P_1, P_2 \uplus \tilde{P}_2', P_3 \uplus P_3'$ where*

$$\tilde{P}_2' = \left\{ \overline{\alpha} : g \Rightarrow f \mid \alpha : f \Rightarrow g \in \tilde{P}_2 \right\}$$
$$P_3' = \left\{ \overline{\alpha} \bullet \alpha \Rrightarrow \mathrm{Id}_f, \alpha \bullet \overline{\alpha} \Rrightarrow \mathrm{Id}_g \mid \alpha : f \Rightarrow g \in \tilde{P}_2 \right\}$$

Proof. Let us define the strict functor of sesquicategory on generators :

$$\begin{aligned} L : \|P\| &\to \|P'\| \\ x \in P_0 &\mapsto x \\ f \in P_1 &\mapsto f \\ \alpha \in P_2 &\mapsto \alpha \end{aligned}$$

By definition, for α in $\tilde{P}_2$, $L\alpha = \alpha$, which is an isomorphism in $\|P'\|$. Besides, for any β and α in P_2^* such that $\alpha \Rightarrow \beta$ is in P_3, then $L(\alpha) = L(\beta)$ in $\|P'\|$ (this relies on the fact that the functor is strict).

Let us consider a weak functor $F : \|P\| \to \mathcal{S}$ with $\mathcal{S}$ a sesquicategory such that $F(\tilde{P}_2)$ is a subset of the isomorphisms in $\mathcal{S}$. We may define the weak functor

$$\begin{aligned} \tilde{F} : \|P'\| &\to \mathcal{S} \\ x \in P_0 &\mapsto Fx \\ f \in \|P'\|_1 &\mapsto Ff \\ \alpha \in P_2^{CC} &\mapsto F\alpha \\ \overline{\alpha} \in \tilde{P}_2'^{CC} &\mapsto F(\alpha)^{-1} \end{aligned}$$

It is extended to 2-cells by Lemma 1.5 and $F(\alpha \bullet \beta) = F(\alpha) \bullet F(\beta)$ (as F is a weak functor). □

Definition 3.6 The *quotient* of a sesquicategory $\mathcal{S}$ by a set Σ of 2-cells of $\mathcal{S}$ is a sesquicategory $\mathcal{S}/\Sigma$ together with a weak *quotient functor* of sesquicategory $Q : \mathcal{S} \to \mathcal{S}/\Sigma$ sending the elements of Σ to identities, such that for every weak functor of sesquicategory $F : \mathcal{S} \to \mathcal{S}'$ sending the elements of Σ to identities, there exists a unique weak functor of sesquicategory $\tilde{F}$ such that $\tilde{F} \circ Q = F$.

Lemma 3.7 *Given three sesquicategories $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$, such that $\mathcal{S}_1$ and $\mathcal{S}_2$ are quotients of the sesquicategory $\mathcal{S}$ by a set Σ of 2-cells of $\mathcal{S}$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic.*

4 Working on the category of morphisms

4.1 Residuation in a sesquicategory

Criterion 4.1 *We suppose fixed a presentation modulo $(P, \tilde{P}_2)$ such that for every α in $\tilde{P}_2$, β in P_2, f, g in P_1^* such that $f.\alpha$ and $\beta.g$ are coinitial and different (resp. $\alpha.f$ and $g.\beta$ are coinitial), there exist α' in $\tilde{P}_2^*$, β' in P_2^* and $R : \beta' \bullet (f.\alpha) \Lleftarrow\!\!\Rrightarrow \alpha' \bullet (\beta.g)$ in P_3 (resp. $R : \beta' \bullet (\alpha.f) \Lleftarrow\!\!\Rrightarrow \alpha' \bullet (g.\beta)$).* FC: expliquer ce qu'est P_2^*, $\tilde{P}_2^*$, $\Leftrightarrow$

We call β' (resp. α') the *residual* of $\beta.g$ after $f.\alpha$ (resp. of $f.\alpha$ after $\beta.g$) and we denote it $\beta.g/f.\alpha$ (resp. $f.\alpha/\beta.g$). Besides, for α in $\tilde{P}_2$, we set that $\alpha/\alpha = \mathrm{Id}$.

Let $\tilde{P}_2^{CC}$ (resp. P_2^{CC}) be the closure of $\tilde{P}_2$ (resp. P_2) under context.

Remark 4.2 The residuation can be extended to α and β in $\tilde{P}_2^{CC}$ and P_2^{CC} respectively and $f = g = \mathrm{id}$ by setting

$$(h_1.\gamma_1.h2)/(h_1.\gamma_2.h_2) = h_1.(\gamma_1/\gamma_2).h_2$$

for γ_1 and γ_2 in P_2^{CC} and h_1 and h_2 in P_1^*.

4.2 Category of morphisms

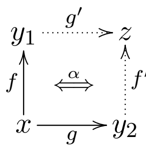
From a presentation modulo of sesquicategory $(P, \tilde{P}_2)$ with $P = (P_0, P_1, P_2, P_3)$, we define the presentation modulo of category $(Q, \tilde{Q}_1)$:

$$\begin{aligned} Q_0 &= P_1^* \\ Q_1 &= P_2^{CC} \\ Q_2 &= P_3^{CC} \\ \tilde{Q}_1 &= \tilde{P}_2^{CC} \end{aligned}$$

The category presented is called the *category of morphisms* associated to $(P, \tilde{P}_2)$. Its composition denoted $\star$ is the composition $\bullet$ in the sesquicategory $\|P\|$. FC: expliquer P_3^{CC}

Assuming that the presentation modulo $(Q, \tilde{Q}_1)$ satisfies all the assumptions of the previous article with the notion of residuation inherited from the presentation of sesquicategory, namely

(i) for every pair of distinct coinitial generators $f : x \to y_1$ in $\tilde{Q}_1$ and $g : x \to y_2$ in Q_1, there exist a pair of cofinal morphisms $g' : y_1 \to z$ in Q_1^* and $f' : y_2 \to z$ in $\tilde{Q}_1^*$ and a relation $\alpha : g' \circ f \Leftrightarrow f' \circ g$ in Q_2



FC: attention que celle-là elle doit maintenant s'exprimer avec le critère d'avant

(ii) there is no infinite path with generators in $\tilde{Q}_1$.

(iii) There is a weight function $\omega_1 : Q_1 \to \mathbb{N}$, and we still write $\omega_1 : Q_1^* \to \mathbb{N}$ for its extension as morphism of category to the category corresponding to the additive monoid $(\mathbb{N}, +)$, such that for every generator $g \in Q_1$ and $f \in \tilde{Q}_1$, we have $\omega_1(g/f) < \omega_1(g)$.

(iv) The presentation $(Q, \tilde{Q}_1)$ satisfies the *cylinder property*: for every triple of coinitial morphisms $f : x \to x'$ in $\tilde{Q}_1$ (resp. in Q_1) and $g_1, g_2 : x \to y$ in Q_1^* (resp. in $\tilde{Q}_1^*$) such that there exists a relation $\alpha : g_1 \Leftrightarrow g_2$, we have $f/g_1 = f/g_2$ and there exists a 2-cell $g_1/f \overset{*}{\Leftrightarrow} g_2/f$. We write α/f for an arbitrary choice of such a 2-cell.

(v) There is a weight function $\omega_2 : Q_2 \to \mathbb{N}$ (which can be extended to any 2-cell of the 2-category generated by Q by $\omega_2(\overline{\alpha}) = \omega_2(\alpha)$ and both horizontal and vertical compositions are sent to addition) such that for every $\alpha : g_1 \Rightarrow g_2$ in Q_2^* and f in Q_1 such that α/f exists we have $\omega_2(\alpha/f) < \omega_2(\alpha)$.

(vi) The presentation modulo $(Q^{\mathrm{op}}, \tilde{Q}_1^{\mathrm{op}})$ satisfies previous assumptions.

then there is an equivalence of category $E_1 : \|Q\| \downarrow \tilde{Q}_1 \to \|Q\| [\tilde{Q}_1^{-1}]$ and an isomorphism of category $I_1 : \|Q\| \downarrow \tilde{Q}_1 \to \|Q\| / \tilde{Q}_1$.

Given f in P_1^*, we denote by $\hat{f}$ its normal form wrt $\tilde{P}_2$.

5 Applying it to sesquicategories

5.1 Definition of the sesquicategory of normal forms

Definition 5.1 The *sesquicategory of normal forms* associated to $(P, \tilde{P}_2)$ is the sesquicategory denoted $\|P\| \downarrow \tilde{P}_2$ whose set of 0-cells is P_0, whose sets of 1-cells and 2-cells are the set of objects and morphisms respectively of $\|Q\| \downarrow \tilde{Q}_1$, whose compositions $\circ_N$, $\bullet_N$ are defined as follows :

$$f \circ_N g = \widehat{f \circ g}$$
$$\alpha \bullet_N \beta = \alpha \bullet \beta$$

The actions $f ._N \alpha ._N g$ where $\alpha : h_1 \Rightarrow h_2$ are obtained from the 2-cell $f.\alpha.g : f \circ h_1 \circ g \Rightarrow f \circ h_2 \circ g$ which is also a morphism in $\|P\|$ and thus induces a morphism $f \circ_N h_1 \circ_N g \Rightarrow f \circ_N h_2 \circ_N g$ in $\|Q\| \downarrow \tilde{Q}_1$ which is also a 2-cell in $\|P\| \downarrow \tilde{P}_2$.

Proof. TODO Montrer que bien une sesquicatégorie ie, composition tout ça ☐

5.2 Theorem

Lemma 5.2 *There is an equivalence of sesquicategory $E_2 : \|P\| \downarrow \tilde{P}_2 \to \|P\| [\tilde{P}_2^{-1}]$.*

Proof. Recall that the functor $E_1 : \|Q\| \downarrow \tilde{Q}_1 \to \|Q\| [\tilde{Q}_1^{-1}]$ is an equivalence of categories. We set the weak functor of sesquicategory :

FC: vérifier que c'est bien un weak functor

$$\begin{aligned} E_2 : \|P\| \downarrow \tilde{P}_2 &\to \|P\| [\tilde{P}_2^{-1}] \\ x \text{ 0-cell} &\mapsto x \\ f \text{ 1-cell} &\mapsto E_1 f \\ \alpha \text{ 2-cell} &\mapsto E_1 \alpha \end{aligned}$$

Let us now prove that this is indeed an equivalence of sesquicategory. It is an isomorphism on 0-cells.

Given two 0-cells x and y in $\|P\| \downarrow \tilde{P}_2$, let us consider $f : x \to y$ in $\|P\| [\tilde{P}_2^{-1}]$. As E_1 is an equivalence of category, there exist $g : x \to y$ 1-cell in $\|P\| \downarrow \tilde{P}_2$ such that there exists an isomorphism ϕ between $E_1 g$ and f in $Q[\tilde{Q}_1^{-1}]$.

FC: vérifier que phi est bien un Sfoncteur faible

Given two 2-cells $\alpha, \beta : f \Rightarrow g$ in $\|P\| \downarrow \tilde{P}_2$ such that $E_2\alpha = E_2\beta$ in $\|P\| [\tilde{P}_2^{-1}]$, then $E_1\alpha = E_1\beta$ in $\|Q\| [\tilde{Q}_1^{-1}]$. As E_1 is faithful, this means that $\alpha = \beta$ in $\|Q\| \downarrow \tilde{Q}_1$

Given two parallel 1-cells f and g in $\|P\| \downarrow \tilde{P}_2$ and $\alpha : E_1 f \Rightarrow E_1 g$ in $\|P\| [\tilde{P}_2^{-1}]$, as E_1 is full, there exists a morphism β in $\|Q\| \downarrow \tilde{Q}_1$ such that $E_1\beta = \alpha$. This proves that E_2 is full on 2-cells. ☐

FC: est-ce que suffit pour conclure ?

Lemma 5.3 *There is an isomorphism of sesquicategory $I_2 : \|P\| \downarrow \tilde{P}_2 \to \|P\| / \tilde{P}_2$.*

Proof. We are going to show that the sesquicategory of normal forms $\|P\| \downarrow \tilde{P}_2$ is a quotient of the sesquicategory $\|P\|$ by $\tilde{P}_2$. Recall that the category $\|Q\| \downarrow \tilde{Q}_1$ is a quotient of the category $\|Q\|$ by $\tilde{Q}_1$, and this gives us a quotient functor $\psi : \|Q\| \to \|Q\| \downarrow \tilde{Q}_1$.

Let us now define the weak functor

$$\begin{aligned}\phi : \|P\| &\to \|P\|{\downarrow}\tilde{P}_2\\ x \in P_0 &\mapsto x\\ f \in P_1^* &\mapsto \hat{f} = \psi f\\ \alpha \in P_2^{CC} &\mapsto \psi\alpha\end{aligned}$$

For any α in $\tilde{P}_2^{CC} = \tilde{Q}_1$, $\phi\alpha = \psi\alpha$ is the identity.

There remains to prove the universal property. □

FC: check that for $\alpha \Rrightarrow \beta$ in P_3, $\phi\alpha = \phi\beta$

TODO : et l'injection de $\|P\|$ dans sa localisée ?

6 Hypothesis in 2-categories

Definition 6.1 A *presentation of 2-category* is $C = (C_0, C_1, C_2, C_3)$ such that (C_0, C_1, C_2) is a presentation of category. It generates a free category with set of objects C_0 and set of morphisms C_1^*. It *generates* a free 2-category with C_0 as set of 0-cells, C_1^* as set of 1-cells and C_2^* as set of 2-cells. The set C_3 is a subset of $C_2^* \times C_2^*$. The set C_3 generates a congruence (symmetric, reflexive, transitive and under both compositions closure) denoted $\overset{*}{\Leftrightarrow}$. The *2-category presented* is the 2-category with set of 0-cells C_0, set of 1-cells C_1^* and set of 2-cells $C_2^*/\overset{*}{\Leftrightarrow}$.

Lemma 6.2 *Given a presentation of 2-category (C_0, C_1, C_2, C_3), it is presented as a Sesquicategory by the presentation of Sesquicategory $(P_0, P1, P_2, P_3)$ with $P_i = C_i$ for $i = 0, 1, 2$ and*

$$P_3 = C_3 \uplus \{X((h_1 \circ g).\alpha, \beta.(g \circ f_1)) \mid \alpha \in P_2, \beta \in P_2, g \in P_1\}$$

with $X((h_1 \circ g).\alpha, \beta.(g \circ f_1)) : (\beta.(g \circ f_2)) \bullet ((h_1 \circ g).\alpha) \Rrightarrow ((h_2 \circ g).\alpha) \bullet (\beta.(g \circ f_1))$ for $\alpha : f_1 \to f_2$ in P_2, $\beta : h_1 \to h_2$ in P_2, $g : t_0(h_1) \to s_0(f_1)$ in P_1.

We consider such a presentation of Sesquicategory.

The first assumption on residuals become :

Criterion 6.3 *We suppose fixed a presentation modulo $(P, \tilde{P}_2)$ such that for every α in $\tilde{P}_2$, β in P_2, f, g in P_1^* such that for every critical pair on words in P_1^* $(f.\alpha, \beta.g)$ (resp. $(\alpha.f, g.\beta)$), there exist α' in $\tilde{P}_2^*$, β' in P_2^* and $R : \beta' \bullet (f.\alpha) \Leftrightarrow \alpha' \bullet (\beta.g)$ in P_3 (resp. $R : \beta' \bullet (\alpha.f) \Leftrightarrow \alpha' \bullet (g.\beta)$).*

Indeed, we set for α in $\tilde{P}_2$

$$\alpha/\alpha = \mathrm{Id},$$

for $\alpha : f_1 \to f_2$ in $\tilde{P}_2$ (resp. P_2), $\beta : h_1 \to h_2$ in P_2 (resp. $\tilde{P}_2$), $g : t_0(h_1) \to s_0(f_1)$ in P_1,

$$\begin{aligned}((h_1 \circ g).\alpha)/(\beta.(g \circ f_1)) &= (h_2 \circ g).\alpha\\ (\beta.(g \circ f_1))/((h_1 \circ g).\alpha) &= (\beta.(g \circ f_2))\end{aligned}$$

and for γ_1 and γ_2 in P_2^{CC} and h_1 and h_2 in P_1^*

$$(h_1.\gamma_1.h2)/(h_1.\gamma_2.h_2) = h_1.(\gamma_1/\gamma_2).h_2$$

That way, we defined

Lemma 6.4 *If the local cylinder property is verified for $\alpha \Lleftarrow\!\!\Rrightarrow \beta$ in P_3 (α and β are in P_2^*) and γ in P_2^{CC} such that*

- *there does not exist f (resp. g) in P_1 such that $\alpha = \alpha'.f$ (resp. $\alpha = g.\alpha'$), $\beta = \beta'.f$ (resp. $\beta = g.\beta'$) and $\gamma = \gamma'.f$ (resp. $\gamma = g.\gamma'$),*
- *when considering α, β and γ pairwise, there is at most one of the pairs that forms and exchange law,*

then the local cylinder property is verified for all α, β and γ in P_2^{CC}.

FC: pas encore ça

Proof. If there are three exchange : closed by exchange

If there are two : closed by exchange and the 3-cell but in a different context.□

7 Product of monoidal categories

TODO

Monoidal category is a 2-category with one object. Any monoidal category admits thus a presentation C as a 2-category where $C_0 = \{*\}$. Given two monoidal categories $\mathcal{C}$ and $\mathcal{D}$, their product $\mathcal{C} \times \mathcal{D}$ in **Cat** is also a monoidal category. Given a presentation C of $\mathcal{C}$ and a presentation D of $\mathcal{D}$, we want to deduce a presentation of $\mathcal{C} \times \mathcal{D}$. Let us consider the presentation of 2-category S where :

$$S_0 = \{*\}$$
$$S_1 = C_1 \uplus D_1$$
$$S_2 = C_2 \uplus D_2 \uplus \{\gamma_{c,d} : d \circ c \to c \circ d \mid c \in C_1, d \in D_1\}$$
$$S_3 = C_3 \uplus D_3 \uplus S_3' \uplus S_3''$$
$$S_3' = \left\{\gamma_{c',d} \bullet (d.\alpha) \Rightarrow (\alpha.d) \bullet \gamma_{c,d} \mid d \in D_1, c, c' \in C_1^* \text{ and } \alpha : c \to c' \in C_2\right\}$$
$$\uplus \left\{\gamma_{c,d'} \bullet (\beta.c) \Rightarrow (c.\beta) \bullet \gamma_{c,d} \mid d, d' \in D_1^*, c \in C_1 \text{ and } \beta : d \to d' \in D_2\right\}$$
$$S_3'' = \{\gamma_{c,d} \bullet d.\eta \Rightarrow \eta.d \mid d \in D_1, c \in C_1^*, \eta : 1 \to c \in C_2\}$$
$$\uplus \{\gamma_{c,d} \bullet \eta.c \Rightarrow c.\eta \mid d \in D_1^*, c \in C_1, \eta : 1 \to d \in D_2\}$$
$$\uplus \left\{\gamma_{c'c,d} \bullet d.\eta.c \Rightarrow \eta.cd \bullet \gamma \mid c' \in C_1^*, c \in C_1, d \in D_1, \eta : 1 \to c' \in C_2\right\}$$
$$\uplus \left\{\gamma_{c,dd'} \bullet d.\eta.c) \Rightarrow cd.\eta \bullet \gamma_{c,d} \mid d' \in D_1^*, c \in C_1, d \in D_1, \eta : 1 \to d' \in C_2\right\}$$

where $\gamma_{c,d}$ is extended to c in C_1^* or d in D_1^* by :

$$\gamma_{c_k \circ \ldots \circ c_1, d} = (c_k.\gamma_{c_{k-1} \circ \ldots \circ c_1, d}) \bullet (\gamma_{c_k,d}.c_{k-1} \circ \ldots \circ c_1)$$
$$\gamma_{c, d_k \circ \ldots \circ d_1} = (\gamma_{c, d_k \circ \ldots \circ d_2}.d_1) \bullet (d_k \circ \ldots \circ d_2.\gamma_{c,d_1})$$

Horizontal composition and actions : tensor product (actions is just a practical notation), and vertcial composition is composition in the monoidal category

cylinder property ? consider all $R : f \Rightarrow g$ in $\mathcal{C}_3$ (or D_3) with f and g paths. It is ok whenever neither f nor g cannot be written as $\mathrm{id}_x . f' . \mathrm{id}_y$. Others have to be checked by hand (although for R with 0-source of length 1, ok)

8 Example