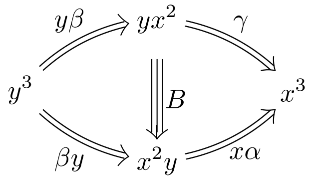
Let consider the presention by rules

$$xy \overset{\alpha}{\Longrightarrow} x^2 \qquad y^2 \overset{\beta}{\Longrightarrow} x^2$$

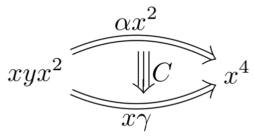
If we complete this presentation by the cubical rule

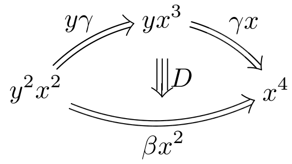
$$yx^2 \overset{\gamma}{\Longrightarrow} x^3$$

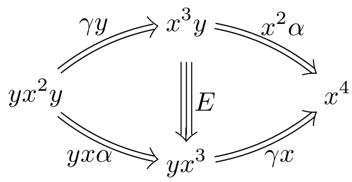
we have confluence diagram



There are new critical pairs







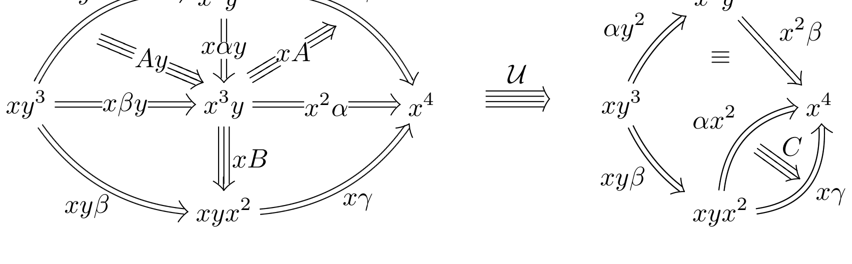
There is no other critical pairs, hence the set $\{A, B, C, D, E\}$ forms an homotopy base of the track category $\Sigma^{\top}$, where $\Sigma_0 = \{\cdot\}$, $\Sigma_1 = \{x, y\}$ and $\Sigma_2 = \{\alpha, \beta, \gamma\}$.

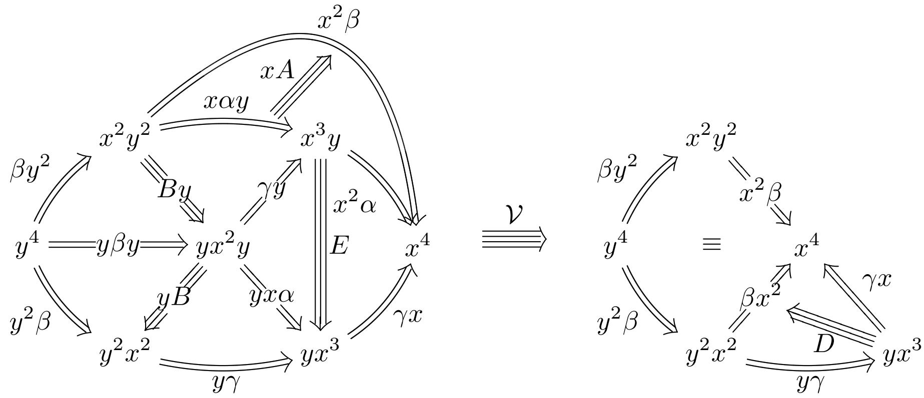
'Removing' B kill γ

There are seven critical triples, their 1-source are

$$xyyy, \quad yyyy, \quad xyyxx, \quad xyxxy, \quad yyyxx, \quad yyxxy, \quad yxxyy.$$

The confluence diagrams for the two critical triple whose 1-source is of length 4 are





'Removing' $\mathcal{U}$ kill C.

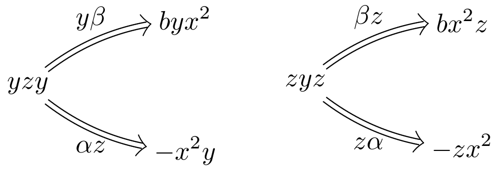
By $\mathcal{V}$, the 3-cell D is related to A, B, E.

There is no 4-cell on the diagonal in order to kill E, hence E still an obstruction to the Koszulness.

Let consider the following quadratic algebra with $a \neq 0, 1$. We orient this two relations using the lexicographic order induced by the alphabetic order $x < y < z$, that is

$$yz \overset{\alpha}{\Longrightarrow} -x^2 \qquad zy \overset{\beta}{\Longrightarrow} bx^2$$

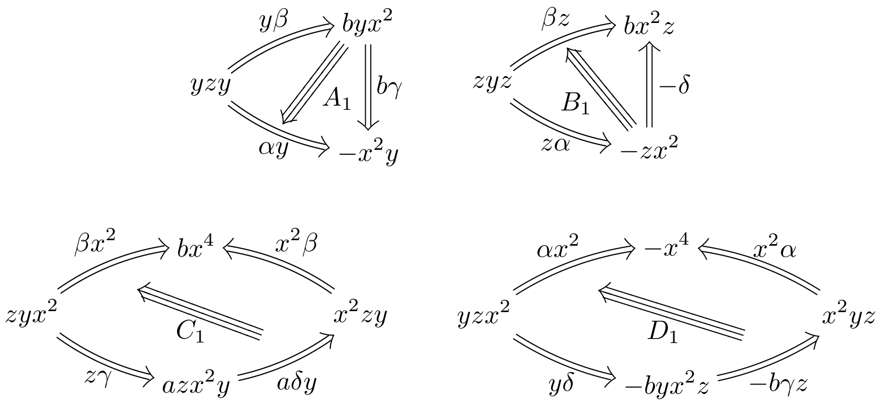
with $b = -1/a$. This rules form two critical pairs :



We complete the polygraph by adding the rules

$$yx^2 \overset{\gamma}{\Longrightarrow} ax^2y \qquad zx^2 \overset{\delta}{\Longrightarrow} -bx^2z$$

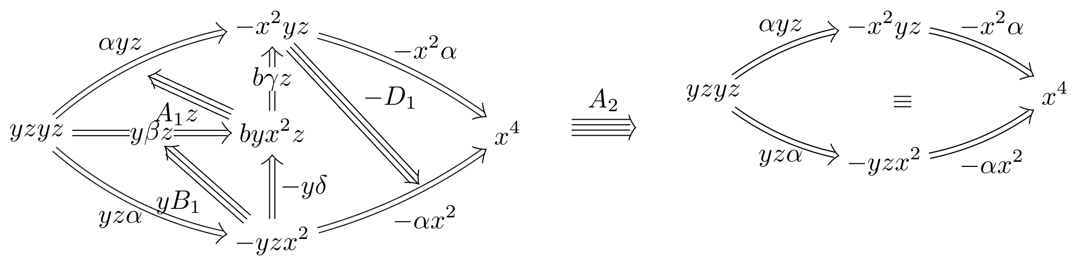
The polygraph is convergent with the following four confluent critical pairs



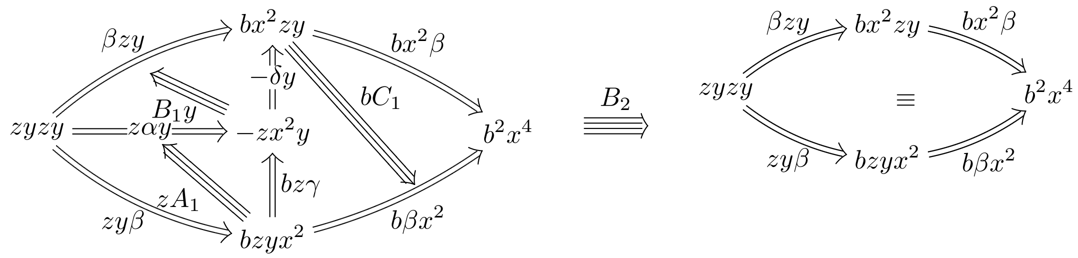
$\{A_1, B_1, C_1, D_1\}$ forms an homotopy base.
'Removing' A_1 kills γ.
'Removing' B_1 kills δ.
There ar four critical triple with 1-sources

$$yzyz,\ yzyx^2,\ zyzy,\ zyzx^2.$$

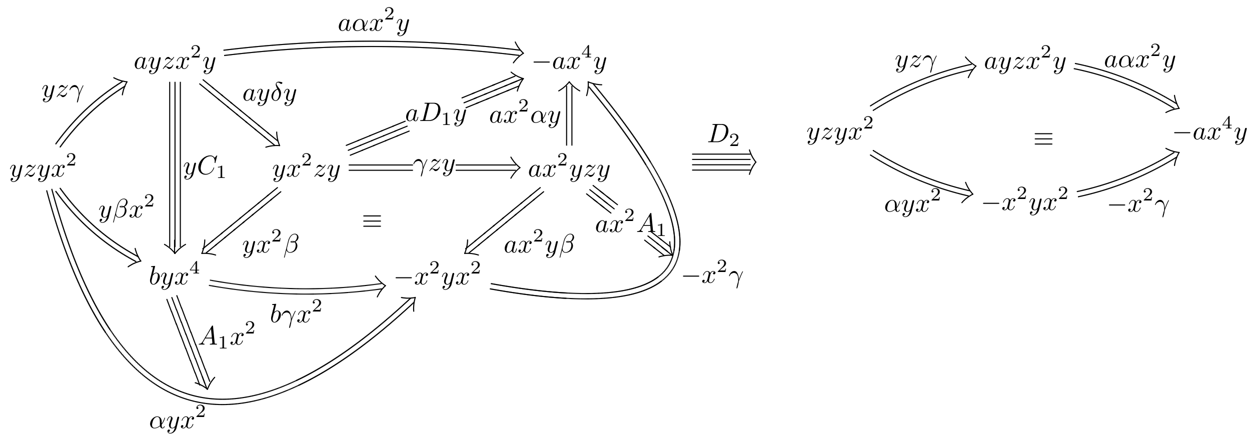
The critical branching on $yzyz$ (resp. $zyzy$) is denoted A_2 (resp. B_2) :

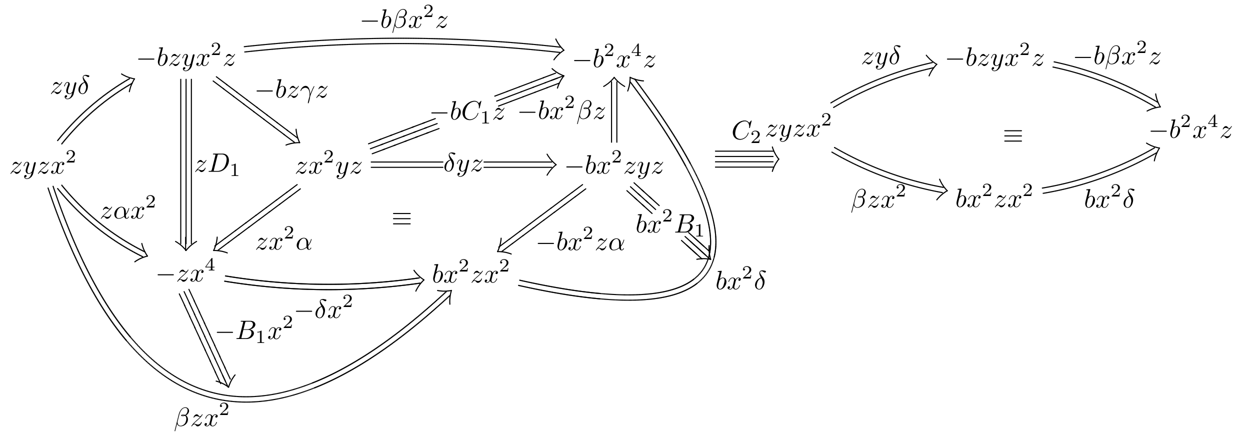


The 4-cell A_2 relates D_1 with A_1 and B_1.



The 4-cell B_2 relates C_1 with A_1 and B_1.
A_2 and B_2 are on the diagonal.
The two other critical triples are not on the diagonal.





−bβx²z
−bzyx²z
−b²x⁴z
zyδ
−bzγz
−bC₁z
−bx²βz
zyzx²
zD₁
zx²yz
δyz
−bx²zyz
C₂
zαx²
≡
bx²B₁
zx²α
−bx²zα
−zx⁴
bx²zx²
bx²δ
−B₁x²
−δx²
βzx²
zyδ
−bzyx²z
−bβx²z
zyzx²
≡
−b²x⁴z
βzx²
bx²zx²
bx²δ