Monomial Algebras

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Introduction

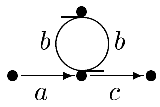
In this survey we study applications of combinatorial methods in the ring theory. Unlike the well-known survey by V.A.Ufnarovski [55], we are more interested in the technique of obtaining these methods and not in results themselfs. Therefore, the themes of this survey is more narrow and it is in more correspondence with the authors interests. The main subject here is the word combinatorial analysis and its applications, and also the notion of the canonical form of element in different calculus.

An element in some algebra can be represented as a linear combination of words, therefore, the problems of the word combinatorial analysis are the most "pure". The study of algebras growth and Hilbert series also can be reduced to the word combinatorial analysis, also as periodicity effects, which appear in the problems of Burnside type (Shirshov height theorem, independency theorem, PI-algebra radical nilpotency and so on).

Monomial algebras, i.e., the algebras, which defining relations are words from generators, are in the center of this survey. The aim of such algebras study is to develope necessary ideas for constructing the combinatorial ring theory. Because of such investigations, the role of uniformly recurrent words and the connection of the combinatorial ring theory with the symbolical dynamics, became more clear.

We study the algorithmical problems in the monomial algebras theory, representations of such algebras, varieties, generated by them, their structurial theory. With the help of monomial algebras we prove and generalize Gelfand-Kirillov height and dimension theorems for arbitrary PI-algebras. The special case is the case of automata algebras, i.e., monomial algebras, which set of nonzero words are regular languages. A regular language can be defined as a set of words, which can be read when moving along some graph edges, which are marked by the letters of the fixed alphabet. The letters of the alphabet corresponds to the generators of the monomial algebra. The graph, which represents a finite automaton, is called the Ufnarovski graph of an automata algebra.

The notion of an automata algebra generalizes the notion of a finitely generated monomial algebra. In proving all results about finitely generated monomial algebras, we in fact use exactly the regularity of the nonzero words language, i.e., the possibility to define it by an oriented graph. Therefore, there is no sense in the study of finitely generated algebras as an individual class, the more so, as there exist automata algebras, wich are not finitely generated. For example, the algebra with generators a, b, c and relations $a^2 = c^2 = ca = ba = 0$, $ab^{2n+1}c = 0, \quad n = 0, 1, 2, \ldots$, is not finitely generated, but is automata. Its graph is



For automata algebras hold many statements, which are wrong for arbitrary (non-monomial) finitely generated algebras. Fore example, the growth alternative – polynomial or exponential, or the implication "the existence of a polynomial identity $\Longrightarrow$ the representability by matrices". The greater part of algorithmic problems have a positive solution for automata algebras. For example, such problems, as the growth computing, the existence of a polynomial identity, the representability, the semisimplicity, the primarity, the existence of Noetherian structure, are algorithmically solvable for automata algebras. In addition, the algebraic properties of automata algebras can be described "geometrically", using the graph language. For individual elements there exist a checking algotithm for nilpotency and a zero divisibility.

At last, the word combinatorial analysis can be used in study of arbitrary (not necessary monomial) algebras. Combinatorial lemmas help in study of the canonical bases, of the normal form of elements. It mainly concerns the height theorems and the independency theorems. The combinatorial reasoning helps to get simple and constructive proofs of such well-known theorems, as Shirshov height theorem, V.A.Ufnarovski and G.P.Chekanu independency theorem, to get a positive solution of Shestakov hypothesis about the local nilpotency of an algebra of the (polynomial) complexity n, in which all words (consisting of generators) of degree n are nilpotent, to get a proof of Razmyslov-Kemer-Braun theorem about the nilpotency of the radical of a finitely generated PI-algebra, and so on.

Which ideas are mainly used here?

At first, it is the idea of lexicographic ordering, the consideration of minimal (maximal) words and nondecreasing objects. It works not only in word study, but also in the consideration of degree vectors (the pump-over lemma) and in comparing word systems.

Then, one of the main instruments here is the study of infinite words, which we call superwords, for shortness. Superwords allow to carry out the nonconstructive combinatorial reasoning with the help of the compactness notion (we can construct an infinite word from the infinite number of finite words).

An exceptionally important is the notion of a uniformly recurrent infinite word W: for each positive integer k there exists $N = N(k)$, such that a finite part of the superword W of length k belongs to each its part of length N. Uniformly recurrent (u.r.) words help, for example, to describe all almost simple monomial algebras (i.e., algebras with nilpotent factors): these algebras are of the type A_W, where A is an u.r. superword (by A_W we denote a monomial algebra, such that all its nonzero words are subwords of the superword W). The proof of this fact is based on the following lemma, which is an analog of the density theorem: for each two subwords $u \neq v$ of the non-periodic u.r. superword W there exist subwords $r, t \subset W$, such that $rut \subset W$, $rvt \not\subset W$.

This result is also used in proving the theorem about the coincidence of the nilradical of a monomial algebra and its Jacobson radical. The nilradical of a monomial algebra is turned out to be equal to the intersection of ideals with monomially almost simple factors (a monomial algebra is called monomially almost simple, if each its factor in respect to ideal, generated by a monomial, is nilpotent). A monomially almost simple algebra is an algebra of the type A_W, where W is an arbitrary u.r. superword.

Superwords also help in describing the weakly Noetherian monomial algebras, in getting short proofs of the theorems about independency and the local nilpotency af Lie algebras, generated by sandwiches. In the superword terms the Ufnarovski ordering can be naturally defined: right superwords constitute a linearly ordered set.

The study of superwords leads to questions, related to the symbolical dynamics. Thus, uniformly recurrent words have a dynamical sense: they are exactly those words, which appear during the investigation of minimal closed invariant (in respect to a shift operator) sets in the superword space with the Hemming metric. It turns out to be that, if the algebra growth function $V(n)$ (i.e., the dimension of the space, generated by words of degree not greater, than n) satisfies the inequality $V(n) < n(n+3)/2$ for some n, then the algebra has a linear growth. Algebras with a “limit” growth function $n(n+3)/2$ can be described in terms of the circle rotations. Namely, all of them, except a countable set, can be constructed as algebras A_W, where $W = \{w_i\}$ is a sequence of 0 and 1, which is defined by irrational numbers $\alpha, \beta \in (0,1) : w_i = f(i+1) - f(i)$, $f(i) = [\alpha i + \beta]$. Dynamical properties of u.r. words (the minimality in respect to the inclusion of a subword system of an u.r. word) are also used in proof of the weak Noetherability criterion.

Thirdly, we often use combinatorial lemmas about periodic words, periodic sequences and the arrangement of periodic parts inside a word. The whole section is dedicated to these small, but useful statements. The pseudoperiodicity – the linear ordering alternative in respect to lexicographic order is studied with special attention. This alternative is based on properties of the shift invariancy of periodic words and on the quasiperiodicity of the word equation $uW = Ws$ solutions. This alternative also clarifies the proofs of the height theorem, of the independency theorem and of the theorem about coincidence of the nilradical and Jacobson radical in monomial algebras. All section 2.1 considers the quasiperiodicity notion.

Forthly, we use, the already mentioned above, automata technique. The using of graphs demonstrates the categorial approach. The objects of the corresponding category are the graph verteces and the morphisms are arrows. To each additive category, such that all its morphisms are denoted by letters of some alphabet, corresponds a subalgebra of the endomorphism algebra of the direct sum of all its objects. If there are no differently denoted paths, which connect two objects (for example, when all morphisms have different denotations), then we get a monomial algebra. There are natural relations between morphisms of marked categories and morphisms of corresponding algebras. All theory of monomial algebras representations is based on this fact. Representations of algebras were studied by many authors, see the survay [7]. If a graph is "good", i.e., it has no "linking" cycles (cycles with a mutual vertex), then the collection of passed arrows uniquely defines the order of passage and the commutativity of morphisms is of no importance here. The algebra, which corresponds to a "good" finite graph, is a PI-algebra. Under the natural constraints, the converse statement is also true.

At last, a simple, but very useful combinatorial statement is the pump over lemma: let A be a PI-algebra, which satisfies a polynomial identity f of degree m. Then each word $w = c_0 v_1 c_1 \dots v_m c_{m+1}$, where c_i are letters, which don't belong to words v_j, can be represented as a linear combination of words $w' = c_{i_0} v'_1 c_{i_1} \dots v'_m c_{i_{m+1}}$, where c_i don't belong to words v'_j and not more, then $m-1$ words v'_j have length greater, than $m-1$ (almost all words v'_j are short). The pumping over is a combinatorial analog of the algebraicity reasoning. This procedure helps to get an easy proof of the Capelli identity, to get constructible estimations of Capelli identity degree and PI-algebra height. It also helps to simplify the proof of Razmyslov-Kemer-Braun theorem about the nilpotency of the radical and A.Chanyshev result about the global nilpotency of a graded PI-algebra, with nilpotent n-th power of homogeneous elements.

An important methodological aspect of the monomial algebras theory is its connection with the canonical form technique. The study of monomial algebras is in essence the "pure" study of a normal form. Therefore the theory of this algebras has applications in the word combinatorial theory, in symbolical computations, in the coding theory, in the ring combinatorial theory and so on. The practice demonstrates that even in those cases, when study don't concern monomial algebras, actually all computations are performed with elements, considered as generators products, i.e., with monomials. The good support of this thesis presents the notion of Groebner base, which is one of the most important methods in the theory of symbolical computations (see also Theorem 1.3).

The theory of monomial algebras helps to study the normal forms. It also can be used in solving problems of the Burnside type (for one of the authors exactly this was the reason to study monomial algebras). Let, for example, A be a monomial algebra with generators a and b and relations $a^2 = b^2 = 0$. It generates the same variety as the algebra of 2×2 matrices. Each nonzero word in A is a subword of the superword $(ab)^\infty$. If in some algebra B an identity f is valid, which is not valid in the algebra of 2×2 matrices, then f is not valid in A. It means that with the help of f we can destroy each period of length greater, than 1. In particular, if n is sufficiently big, then the word $(a'b')^n$ is linearly representable by words, which contain ${a'}^2$ and ${b'}^2$. So, we created the squares! Let $c = {a'}^2$. If we created a word with a sufficiently great occurance of c, then, by using the height theorem (see the circulation lemma 2.79), we shall create the power $(cu)^k$, and then, using f, the square c^2, i.e., ${a'}^4$. This method, explained in Chapter 2, leads to the proof of the boundedness of a PI-algebra height over a set of words with degrees not greater, than the algebra complexity.

The obtaining of the exponential estimation for the height clarifies the analogy between the structural and the combinatorial reasoning. Therefore, the concluding part of Chapter 2 is organized, as the proof of this estimation.

The structural theory, which helped to obtain basic results in the assotiative ring theory, by the reason of its effectivity, decelerated the developing of the combinatorial methods, maybe more laborious, but constructive. For this reason, several fundamental results in the ring theory (the radical nilpotency theorem for PI-rings, for example) don't have direct proofs. It causes additional difficulties in translating results to a different situation and often make it impossible to obtain reasonable estimations. In obtaining combinatorial proofs one must understand, which combinatorial objects correspond to the structrural theory and how the word combinatorial theory reflects the structural properties.

In creating the structural theory the Burnside type problems played a special role. This problems are important in the combinatorial theory also. They help to clarify the correspondence between structural and combinatorial reasoning.

The main aspect of the structural reasoning is the consideration of the semisimple (semiprime) part and the reduction of the problem to it by the factorization with respect to the radical. It turns out to be that the prime part is a monomial algebra, which corresponds to a periodic word. It generates a matrix variety $\mathrm{Var}(\mathbb{M}_n)$, where n is the period length. If the word has a periodic part of a superword u^∞ of a period, greater, than n, and an identity f is valid, which is not valid for $n \times n$ matrices, then we can destroy the period with the help of f. By this procedure, we can produce a word, lexicographically smaller, than u. Then we can use the technique of selected sets of words, which allows to make constuctive "the reasoning in the semisimple part". (A word collection is called selected, if the quotient algebra by the ideal, generated by this collection, is nilpotent. The property of being selected is independent with the respect to the radical). It helps to create a periodic part with a period smaller, than n. At last, the pumping over is the property similar to the algebraicity. To the unitary completeness of the variety, generated by the matrix algebra, corresponds the deletion and addition lemma. The consideration of irreducible modules corresponds to the consideration of a superword, which is linearly non-representable by smaller words. Let us demonstrate all this on the example.

Theorem. *The set of lexicographically non-diminishable words in the PI-algebra A has a bounded height over the set of words with degrees not greater, than the A complexity.*

Proof. Let m be the minimal degree of A identities and $n = \mathrm{PIdeg}(A)$ be the A complexity. As A has a bounded height over the set of words of degree not greater, than m, then it is enough to prove that, if $|u|$ is a noncyclic word of length not greater, than n, then the word u^k is a linear combination of lexicographically smaller words, if k is sufficiently big.

1. Let us consider the right A-module M, which is defined by the generator v and by relations $vW = 0$, where $W \prec u^{\infty/2}$ (i.e., W is smaller, than some power of u). (By $u^{\infty/2}$ is denoted the infinite to the right word with u as the period, by "$\prec$" is denoted the relation of the lexicographic ordering). The correspondence $t : vs \to vus$ correctly defines the endomorphism of M, hence M can be considered, as an $A[t]$-module. Our aim is to prove that $Mt^k = 0$ for some k.

2. If $Mt^k \in M \cdot J(\mathrm{Ann}\, M)$, where $J(\mathrm{Ann}\, M)$ is the Jacobson radical of the annihilator, then $Mt^{lk} \in M \cdot J(\mathrm{Ann}\, M)^l$ and, by Braun theorem about the radical nilpotency, $Mt^{lk} = 0$, for l sufficiently big. (Using t centrality in $A[t]$, we can get along with Amitzure theorem about the local nilpotency of the radical). Hence, passing to the module M over the quotient algebra $B = A[t]/J(\mathrm{Ann}\, M)$, we can assume that $J(\mathrm{Ann}\, M) = 0$.

3. Using the primary factorization, we can reduce the proof to the case, when M is an exact module over a semiprime ring B.

4. Elements from the center $Z(B)$ don't have annihilators, hence we can localize, with respect to them, and, by considering an algebraic extension of $Z(B)$, come to the case, when M is the matrix algebra over a field. The matrices dimension here is not greater, than n.

5. By applying the construction 2 from 1.4, we get a minimal nonzero right superword $vu^{\infty/2}$ and, hence, get a contradiction to Corollary 2.39 of the independency theorem (see 2.1.4). □

Let us make some remarks. The module M is very similar to an irreducible module. There is a parallelism of reasonings, which are related to the consideration of such modules and to the consideration of words, linearly nonrepresentable by smaller ones. The module M can be defined by left superwords, which have the period u in infinity; t is a shift operator here. A periodic word is an eigenvector for the operator t with the eigenvalue u. The belonging to the Jacobson radical means, besides all other things, in addition, the absence of nonzero eigenvalues. The consideration of left superwords allows to prove the nilpotency of $J(A[t])$. Let us note that the analogy between the structural and the combinatorial reasonings is not clear enough and is of need of further clarifying.

Let us describe in a short form the situation with bases af algebras. The height theorem means that each word is representable as a linear combination of piecewise periodic words, i.e., words of the type

$$v_1^{k_1} v_2^{k_2} \dots v_h^{k_h}, \quad \text{where } h \leq H,\ H \text{ is a constant}$$

The local finiteness of PI-algebras and the boundedness of Gelfand-Kirillov dimension are consequences of this result, because the number of representations of N, as a sum $k_1|v_1| + \cdots + k_h|v_h|$, where $h \leq H$, has the order N^{H-1}.

The further problems are as follows: which words can be taken as v_i and what is the structure of the power vector? As v_i we can take the set words, which degrees are not greater, than the complexity (also we can take any collection of alements, for which the Kurosh problem holds). Now, let us consider the power vector $(k_1, \dots, k_h)$. The essential height, i.e., the number of positions, such that k_i can be simultaneously unbounded, equals to the Gelfand-Kirillov dimension for representable (and, by the A.R.Kemer [24] result, for relatively free) algebras. Nevertheless, even in the representable case, the set of power vectors can have a bad structure, i.e., it can be the complement to the set of solutions of an exponent-polynomial system of Diophantine equations.

Let us present an approximate contents of chapters. Chapter 1 is essentially dedicated to infinite words and u.r. words. Lemmas, proved here, are repeatedely used in what follows. We also consider here growth problems in words and algebras. In particular, we construct a patological algebra, which growth function sometimes is smaller, than $\varphi(n) = n(n+3)/2 + \lambda(n)$, where $\lambda(n)$ is a function, converging (with an arbitrary slowness) to infinity, $ln(n)$ for example, and sometimes is greater, than $\psi(n) = e^{o(n)}$, $\psi(n) = e^{\sqrt{n}}$, for example. Also, we present the description of algebras with the "limiting" slow growth $n(n+3)/2$.

In Chapter 2 we apply the word combinatorial analysis to problems of the Burnside type and to normal bases. The aim here is the obtaining the purely combinatorial (hence, constructive) proofs of already known results and also the obtaining explicit estimations. We give much attention to periodic words properties. Of several theorems, proved here, we must mention the Ufnarovski-Chekanu independency theorem and Chirshov height boundedness theorem.

The relation between radicals and superwords is used here for proving the nonexistence of algebras, which growth functions are between linear ones and the function $n(n+3)/2$. In 2.3 we consider regular words and their applications to Lie algebras: we prove the nilpotency of a subalgebra, generated by sandwiches, and prove the height theorem for Lie algebras with a sparse identity.

In Chapter 3 we prove the theorem about the coincidence of the nilradical and Jacobson radical in a monomial algebra. Jacobson radical is described in terms of u.r. words.

In Chapter 4 it is proved that, if two finitely generated monomial algebras without units are isomorphic, as algebras, then they are isomorphic, as monomial algebras, i.e., there exists an isomorphisms, which maps generators into generators. Also here we study Baire radical of monomial algebras. We prove that the Baire order (i.e., the ordinal that marks the stabilization beginning in the Baire radical construction) can be equal to any ordinal α. For 2-generated monomial algebral α is countable and can be any countable ordinal.

In Chapter 5 we study automata algebras. The main attention here we give to algorithms and to the dependence of algebraic properties of an automata algebra from the "geometric" properties of its graph. Almost all structural properties of an automata algebra (semisimplicity, semiprimarity, growth function, Noetherability and so on) are algorithmically recognizable. Also we present algorithms for checking the nilpotency and the zero divisionability of individual elements.

In Chapter 6 we study finite-dimensional representations of monomial algebras. We consider "tame" and "wild" algebras, i.e., algebras, which representations can or cannot be described. Tame monomial algebras are 1-generated and 2-generated algebras with zero multiplication only. The description of all irreducible representations of an automata PI-algebra can be reduced to the same problem for the algebra A_{u^∞}. If an automata algebra is not a PI-algebra, then it is wild.

The necessary condition of the representability is the validity of the height theorem. In this case, there exists h, such that any word in the algebra can be written as $u_1^{k_1}u_2^{k_2}\dots u_l^{k_l}$, where $l \leq h$ and $\{u_i\}$ is the set of words of fixed length. The necessary and sufficient condition of the representability can be formulated, as a condition on the set of vectors $\bar{k} = \langle k_1, \dots, k_l\rangle$. It means that the set of $\langle k_1, \dots, k_l\rangle$, such that $u_1^{k_1}\dots u_l^{k_l} = 0$, must be defined by a system of exponential Diophantine equations. We have the following theorem, as a corollary.

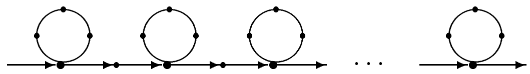
Theorem. *If a monomial algebra is representable over a field of zero characterictic and has Gelpand-Kirillov dimension* 1*, then it is automata and its Hilbert series is rational.*

Moreover, we construct an example of a monomial algebra (or a semigroup), which is representable over a field of positive characteristic, but is not representable over a field of zero characteristic. Also we construct a monomial algebra, which is representable over a field of zero characteristic and has a transcendental Hilbert series.

In Chapter 7 we study varieties, generated by an arbitrary set of monomial algebras. Such varieties constitute a rather extensive class: for example, the varieties, generated by the matrix algebra and by the algebra of upper triangular matrices, can also be generated by automata algebras. But not all varieties can be realized in such way, for example, the variety, defined by the identity $x^n = 0$, and the variety, generated by Grassmann algebra, cannot be generated by monomial algebras. The main result of this chapter is the following theorem.

Theorem. *Each variety, generated by an arbitrary set of monomial (not necessary automata or finitely generated) algebras, can be generated by one automata algebra.*

For proving this theorem, the technique of formal power series is used. Then we give in graph terms the classification of varieties, generated by automata algebras. Such variety is a finite union of varieties of automata algebras, defined by graphs of the following type



(stright paths between loops and loops themselfs have arbitrary length), which satisfy the following condition: each arrow, which has a common vertex with a loop, is marked by the letter, which is not used in other places.

Corollary. *Each variety of monomial algebras is generated by a finitely defined algebra.*

Therefore, letters can coincide only on inner arrows of stright paths, which connect the loops. For proving this theorem, a rather cumbersome graph technique is used, which allows, by the means of "elementary" operations, to simplify graph without changing the generated by it variety (or consecutive varieties are included in one another).

As a consequence of this theorem and also, as a consequence of that fact that each variety, generated by an automata algebra with a cyclic graph, coincides with a variety, generated by the matrix algebra, we get the classification of unitary closed varieties of monomial algebras.

Theorem. *Each unitary closed variety, generated by monomial algebras, coincides with a finite union of varieties, which identity ideals are finite products of identity ideals of matrix algebras.*

Roughly speaking, in the case of algebras with unit, there are no other varieties of monomial algebras, except varieties of matrix algebras and of their semidirect products.

In the concluding part of this work we study varieties of associative algebras with Lie nilpotency identity of a fixed index. We concentrate on problems of algorithmical solvability in these varieties. We demonstrate the algorithmical solvability of the problem about the realizability of the Lie nilpotency identity of a fixed index in a finitely generated algebra, in which the word problem is solvable. Also we prove that in the variety of Lie nilpotent algebras the word problem can be solved by the algorithm, which use Groebner bases technique.

In Appendix A we study nonassociative rings (in particular, alternative and Jordan rings). The main attention here is given to the representability problems, to the height boundedness and to Kurosh problems. Many results in the theory of associative rings can be transfered to the nonassociative case. The possibility of such transfere and the formulation of the "asimptotical nearness to the associativity" criterion clarify the associative theory itself.

At last, Appendix B is dedicated to problems of Burnside type for semirings (in semirings the subtraction operation is not defined). We prove here a generalization of Nagata-Higman theorem for semirings with a noncommutative addition.

Our exposition demands only minimal preliminary knowledge, mainly related to the theory of PI-algebras. Several important combinatorial topics is not considered here. Our work with identities has the qualitive, mainly asimptotical character, we don't work with actual identities. We don't examine the theory of central (Razmyslov) polynomials (there is a good exposition of this theory in I.V.L'vov preprint [35]) and also the Young diagram theory, related to the symmetrical group representations. We don't even mention the supertechnique, invented by A.R.Kemer, which allows to reduce the study the infinitely generated algebras identities to the case of finitely generated superalgebras. We don't examine the homological technique and the diamond lemma. The 2-word method is not considered, and the Lie case and the case of algebras near to associative, are examined only in brief.

The big part of this survey is dedicated to well known results, which either were published before, or are the mathematical folklore with unclear authorship. In any case we beg a pardon from those authors, whom we didn't mention in this text.

In preparation of this work, we used the V.A.Ufnarovski survey [55] and J.Okninski book [87]. The idea of using the uniformly recurrent words appeared as a result of discussions with M.V.Sapir. The representability criterion for monomial algebras is a variant of D.Anick ideas. T.Gateva-Ivanova gave the definition of the primary superword, and our exposition of the theory of the monomial algebra radical is based on the results of the joint work. To our regret, we couldn't have her as a co-author, because of the time shortness and the organizational problems. We hope, however, for further collaboration. S.V.Pchelincev participated in discussions of problems about the height theorem and about the nonassociative case. We are grateful to E.I.Zelmanov and A.V.Mihalev for posing some problems and the moral support. Authors are grateful to L.A.Bokut',

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1 The word combinatorial theory and its applications

In the combinatorial reasoning in the ring theory we mainly work with words, i.e., with the representation of elements via generators. The word technique uses monomial algebras, in which the defining relations constitute the set of word equal to zero. Several problems in the ring theory, the investigation of Hilbert series for example, can be reduced to the monomial case. Let us give some definitions.

1.1 Definitions and notation

All algebras, unless otherwise stipulated, are considered to be finitely generated (f.g.) with a fixed set of generators. Let $\Phi \ni 1$ be an associative and commutative ring. By $\Phi\langle x_1, \ldots, x_s\rangle$ will be denoted a free associative Φ-algebra with generators $x_1, \ldots, x_s$. By $A\langle a_1, \ldots, a_s\rangle$ will be denoted an arbitrary Φ-algebra with a fixed set of generators $a_1, \ldots, a_s$. A word or a monomial from the set of generators $\mathcal{M}$ is an arbitrary product of elements in $\mathcal{M}$. The set of all words constitutes a semigroup, which will be denoted by $\mathrm{Wd}\langle\mathcal{M}\rangle$. The order $a_1 \prec \cdots \prec a_s$ generates the lexicographic order on the set of words: of two words those is greater, which first symbol is greater, if the first symbols coincide, then the seconds are compared, then thirds and so on. Two words are incomparable, only if one of them is the beginning of another. Let us note that a family of the pairwise incomparable words constitutes a linearly ordered set. By a word in an algebra we understand a nonzero word from its generators $\{a_i\}$. An algebra A is called monomial, if it has a base of defining relations of the type $c = 0$, where c is a word from $a_1, \ldots, a_s$. Obviously, a monomial algebra is a semigroup algebra. More precisely, it coincides with the semigroup algebra over the semigroup of its words.

By $|v|$ will be denoted the length of a word v. By $\|x\|$ will be denoted the homogeneity degree of an element x. By $u \subset v$ will be denoted the occurence of a word u in a word v. By $(W)_k$ will be denoted the beginning subword of the word W of length k. By $\langle M\rangle$ will be denoted the Φ-module, generated by a set M. An element x is called linearly representable by a set M, if $x \in \langle M\rangle$. The set of words, which are linearly nonrepresentable by smaller words, is linearly independent and constitutes a normal base of the algebra. By $\mathrm{id}(M)$ will be denoted the bilateral ideal, generated by a set M.

A word w is called n-encountered in a word W, if W has n nonoverlapping occurrences of the word w. A set of words $\mathcal{U}$ is called k-encountered in a word W, if each word $u \in \mathcal{U}$ has k nonoverlapping occurrences in W.

A word is called nonimprovable, if it cannot be represented as a linear combination of lexicographically smaller words.

A word u is called cyclic, if for some $k > 1$, $u = v^k$, otherwise it is called noncyclic or nonperiodic. Words u and v are called cyclically conjugate, if for some words c and d $u = cd$, $v = dc$. The cyclic conjugacy relation is an equivalency relation.

Let us enumerate the usual notations: m be the minimal degree of identity, which holds in A; $n = \mathrm{PIdeg}(A)$ be the complexity of A, i.e., the maximal positive integer k, such that all identities in A hold in the matrix algebra of order k; p be the minimal degree of an identity of the complexity n, which holds in A; by s or l we shall denote the number of generators. By $M^{(k)}$ will be denoted the ideal $\mathrm{id}\{m^k, m \in M\}$. Λ is the empty word, $|\Lambda| = \|\Lambda\| = 0$.

Let us note the following useful statement, which is proved with the word combinatorial technique.

Proposition 1.1 *Let A be a f.g. graded algebra, M be a finite set of homogeneous elements, such that, for all k, the quotient algebra $A/M^{(k)}$ is nilpotent. Then each quotient algebra $A' = A/I$, in which all projections of elements from M are algebraic, is finite-dimensional.*

Proof. Let k be the maximum of degrees of those polynomials, which annihilate elements in the M projection. Let $A/M^{(k)}$ is nilpotent of index l. It means that all words in A of the length not smaller, than l, are linearly representable by words from A generators and those elements $m \in M$, which has m^k, as its subword. Hence, projections of words of length l are linearly representable by projections of words of smaller length. $\square$

1.2 The basic properties of PI-algebras

An identity in algebra is a noncommutative polynomial, which is identically zero on the set of algebra elements. An algebra with an identity is called a PI-algebra. If an identity holds, then all identities, which can be produced from it by substitutions and also by left and right products by arbitrary polynomials, hold also. In the case of zero characteristic, each system of identities is finitely based (unlike the case of relations). This was proved by A.R.Kemer. A.V.Grishin proved that we can manage by substitutions only. Examples of identities are: the commutativity identity $[x, y] = xy - yx$, the standart identity of degree n

$$\mathrm{St}_n = \sum_\sigma (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)},$$

the Capelli identity

$$\sum_\sigma (-1)^\sigma y_0 x_{\sigma(1)} y_1 \dots x_{\sigma(n)} y_n,$$

which holds in any $(n-1)$-dimensional algebra. It is more difficult to prove that St_{2n} holds in each matrix algebra of order n (Levitski-Amizur theorem). The Hall identity $[[x,y]^2,z]$ holds in the algebra of 2×2 matrices (the square of a matrix with zero trace is a scalar matrix). The identity $[[x,y],z]=0$ holds in Grassmann algebra (an even element belongs to the center and the commutator of two odd elements is even). The class of algebras, which satisfy some set of identities, is called a variety; the variety, which corresponds to the set of identities, which hold in the algebra A, is denoted by $\mathrm{Var}(A)$.

If the number of elements in the ground field is greater, than the degree of an identity f, then with f hold all its homogeneous, in respect to each variable, components. If a variable x occurs in an identity f in degree n, then f is linearizable. Let t be the set of variables, distinct from x. Then the polynomial

$$\tilde{f}=f(t,x_1+\cdots+x_n)-\sum_i f(t,x_1+\cdots+\widehat{x}_i+\cdots+x_n)+$$
$$+\sum_{i<j} f(t,x_1+\cdots+\widehat{x}_i+\cdots+\widehat{x}_j+\cdots+x_n)-\cdots+(-1)^{n-1}\sum_k f(t,x_k)$$

is polylinear and symmetric for $x_1,\ldots,x_n$. It is called a linearization of f. (The symbol $\widehat{\ }$ means, as usual, that the corresponding term is omited.) The complete linearization is a polylinear polynomial. If the ground field characteristic is zero or is greater, than f degree, then f is equivalent to its complete linearization. Example: the complete linearization of the identity $[[x,y]^2,z]$ is the identity $[[x_1,y_1]\cdot[x_2,y_2],z]+[[x_1,y_2]\cdot[x_2,y_1],z]+[[x_2,y_1]\cdot[x_1,y_1],z]+[[x_2,y_2]\cdot[x_1,y_2],z]$.

A variety is called unitary closed, if the operation of unit adding preserves the membership in this variety. A matrix algebra generates an unitary closed variety and a nilpotent algebra doesn't.

The set of all identities of some algebra constitutes a fully characteristic ideal, or T-ideal, in the free algebra $K\langle X\rangle$ (here K is the ground field and $\langle X\rangle$ is the countable set of generators). The quotient algebra $K\langle X\rangle/T$, in respect to a T-ideal T, is called a relatively free algebra, or a free algebra of the variety $\mathfrak{M}$, defined by the identities ideal T. The Jacobson radical of this algebra coincides with the set of identities in the matrix algebra of order n. The number n is called a complexity or a polynomial degree of the variety (or the T-ideal) $\mathfrak{M}$ and is denoted by $\mathrm{PIdeg}(\mathfrak{M})$. Varieties of complexity 1 are called non-matric. In each non-matric variety the identity $[x,y]^k$ holds. In a variety of complexity n some power of the standart identity of order $2n$ holds. If an algebra is finitely generated, then in it hold a standart identity of some order and Capelli identity (this statement is consequence of Razmyslov-Kemer-Braun theorem about the radical nilpotency in a finitely generated PI-algebra).

Let f be a polylinear identity of degree n. Then the symmetric group S_n acts from the left, by permuting the variables, on f consequences of degree n, hence, they constitute a left module over its group algebra. Irreducible submodules correspond to Young diagrams. The standart identity corresponds to the diagram-column, the symmetic identity corresponds to the diagram-row. The connection between identities and diagrams is enigmatic. It is known that, if an identity f holds, then all identities, which Young diagrams has a sufficiently big square, hold also. Each type of irreducible modules occures with multiplicity, which equals its dimension. One-dimensional modules correspond to diagram-rows and diagram-columns.

Let $\tau \in S_n$, then we can relate to the identity $f = \sum_\sigma a_\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$ the identity $f^\tau = \sum_\sigma a_\sigma x_{\sigma(\tau(1))} \dots x_{\sigma(\tau(n))}$. If f is not the symmetric or the standart identity, then f^τ is not equivalent to f for some τ. In other words, if we identify polylinear polynomials of degree n and elements in the group algebra of the symmetric group, then the set of all polylinear identities of degree n always constitute a left ideal in this group algebra (but not always a right ideal). (Under a permutation of positions in all monomials in the identity we not always get a consequence of this identity). The same is true for sparse identities. Nevertheless, for sufficienly big n the set of all polylinear identities of degree n contains a bilateral ideal of the group algebra of S_n. The proof of this statement uses Regev theorem (the growth of codimension of a T-ideal is not greater, than exponental) and the dimension formula for an irreducible representation of a symmetric group, which corresponds to a rectangular Young diagram. It turns out to be that the growth of the dimension of a representation is greater, than the growth of a T-ideal codimension. Hence, the bilateral ideal, which corresponds to this Young diagram, belongs to the T-ideal.

The above reasoning holds for sparse identities also, i.e., for identities of the type

$$\sum_{\sigma \in S_n} \alpha_\sigma y_0 x_{\sigma(1)} y_1 \dots x_{\sigma(n)} y_n = 0.$$

(only x_i are permuted and y_j are stationary).

As a positions permutation (i.e., the right action of the symmetric group) realizes an isomorphism of left modules and doesn't change Young diagrams, then we have the following proposition.

Proposition 1.2 *Let T be an arbitrary T-ideal. Then there exists the set of coefficients $\{\alpha_i\}$, such that for each permutation $\tau \in S_n$ and all x_i, y_i the following equality holds modulo T*

$$\sum_{\sigma \in S_n} \alpha_\sigma y_0 x_{\sigma(\tau(1))} y_1 \dots x_{\sigma(\tau(n))} y_n = 0.$$

Let $\mathfrak{M}$ be an algebra variety. A_l be a relatively free l-generated algebra in $\mathfrak{M}$. For A_l some additional identities can be valid (if $l = 1$, for example, then A_l is commutative). l is called the base rank of $\mathfrak{M}$, if $\mathfrak{M} = A_l$. For example, the base rank of $\mathbb{M}_n$ equals 1, if $n = 1$, and equals 2, if $n > 1$. The base rank of a Grassmann variety $\mathbb{G}$, generated by the identity $[[x, y], z] = 0$, equals infinity. (This identity doesn't hold in the algebra of 2×2 matrices, and all primary algebras in $\mathbb{G}$ are commutative. Hence, the commutator ideal belongs to the nilradical, i.e., to the intersection of all primary ideals. In the finitely generated case, by Braun theorem, the radical is nilpotent of a bounded index. For an infinitely generated Grassmann algebra it is not so).

If $l_1 > l_2$, then $\mathrm{Var}(A_{l_1}) \supseteq \mathrm{Var}(A_{l_2})$. For which l there is the strict inclusion $\mathrm{Var}(A_l) \supset \mathrm{Var}(A_{l+1})$? The set of such l is called the transition set. Our knowledge about such sets is poor. In the finitely based case (if the ground field is of zero characteristic) we can prove the countability of the set of all transition sets (which correspond to all possible varieties). Hence, not each set $\mathcal{P} \subset \mathbb{N}$ can be a transition set. (We can construct a non-transition set in the following way: it is known that the standart identity holds in each f.g. algebra, and an algebra, in which the standart identity holds, has a finite base rank. Therefore, if the set $\mathcal{P}$ is too sparse (i.e., it has too big segments of the rank constancy), then one of these segments (in dependence on the variety degree) will give the validity of the standart identity and the finiteness of the base rank).

1.3 Some constructions

1.3.1 The representation of monomial algebras

Let $\mathcal{M} \subset \mathrm{Wd} < \{\S_\rangle\} >$. By $\widehat{\mathcal{M}}$ will be denoted the set of all subwords in $\mathcal{M}$. Then $\mathrm{Wd}\langle\{x_i\}\rangle\backslash\widehat{\mathcal{M}}$ is an ideal in the semigroup $\mathrm{Wd}\langle\{x_i\}\rangle$. It generates the ideal $I_{\mathcal{M}}$ in the algebra $\Phi\langle\{x_i\}\rangle$. The corresponding quotient algebra is denoted by $A_{\mathcal{M}}$. Let W be an ininite word. Then A_W is the algebra, such that all its relations are of the type $v = 0$, where v is a word, which is not a subword in W. If $\{W_i\}$ is a set of words, then A_{W_i} is an algebra, such that all its relations are of the type $v = 0$, where v is a word, which is not a subword in any $\{W_i\}$.

1.3.2 The construction of a monomial algebra with the same growth function, as the given algebra

Theorem 1.3 *For each algebra A there exists a monomial algebra $\widehat{A}$ with the same growth function.*

Proof. Let put in order the generators $x_1 \prec \cdots \prec x_s$, $a_1 \prec \cdots \prec a_s$. Then let us put in order the set of words from $\{x_i\}$ and $\{a_i\}$: at first, by the length, and then lexicographically. Let us consider the epimorphism $\pi : \Phi\langle x_1, \ldots, x_s\rangle \to A\langle a_1, \ldots, a_s\rangle$, $x_i \to a_i$. The subset in $\mathrm{Wd}\langle\{x_i\}\rangle$, generated by words, which projections can be represented as linear combinations of projections of smaller words, is an ideal in the semigroup $\mathrm{Wd}\langle\{x_i\}\rangle$. It generates the ideal I_A in the algebra $\Phi\langle\{x_i\}\rangle$. Let us denote by $\widehat{A}$ the corresponding quotient algebra. It is a monomial algebra with the same growth function, as the growth function of A.

Therefore, the study of growth functions, can be reduced to the monomial case. However, let us note that $\widehat{A}$ can be not finitely defined, even, if A is finitely defined. V.A.Ufnarovski, for example, constructed a finitely defined algebra of the intermediate growth [49]. The corresponding monomial algebra has the intermediate growth also, but cannot be finitely defined, because the Gilbert series of a finitely defined monomial algebra is rational and its growth function satisfies a recurrent relation and has a polynomial or an exponental growth. Therefore, in the finitely defined situation there is no reduction of the growth function description to the monomial case. □

1.4 Superwords in algebras

The main part of combinatorial results in this chapter is based on the consideration of infinite words or superwords. In this subsection are gathered the basic technical facts and constructions, related to superwords in algebras.

Definition 1.4 A superword is a word, which is infinite in both directions. A word, which is infinite to the left, is called a left superword, a word, which is infinite to the right, is called a right superword.

Definition 1.5 By u^∞ will be denoted a superword with the period u, by $u^{\infty/2}$ will be denoted a right (left) superword, which begins (terminates) with the word u.

It will clear from the context, which superword is under the consideration, left or right, therefore we don't introduce a special notations. The writing $u^{\infty/2} \cdot s \cdot v^{\infty/2}$ means, for example, that $u^{\infty/2}$ is a left superword and $v^{\infty/2}$ is a right one.

Right superwords (unlike finite words, where exist incoparable elements) constitute a linear ordered set in respect to the left lexicographic ordering, the same is true for left superwords in respect to the right lexicographic ordering. We shall mainly deal with superwords and right superwords.

We cannot speak about the value of a superword in an algebra, but can speak about its equality or nonequality to zero, and, in some cases, about its linear dependence.

Definition 1.6 a) A superword W is called zero superword, if it has a finite zero subword, and it is called nonzero superword, if it has no finite zero subwords.

b) A finite set of right subwords $\{W_i\}$ is called linear dependable, if there exist $\{\lambda_i\}$, not all of them equal to zero, such that the following relation holds for k sufficiently big:

$$\sum \lambda_i (W_i)_k = 0$$

(by $(W)_k$ is denoted a beginning segment of W of length k).

c) Let M be a right A-module, W be a right superword in A and $m \in M$. We say that $mW \neq 0$, if $\forall k \quad m(W)_k \neq 0$. Otherwise, $mW = 0$.

d) Let M be a right A-module, $\{W_i\}$ be a finite set of right superwords in A and $\{m_i\} \subseteq M$. We say that $\sum m_i W_i = 0$, if for all sufficiently big k, $\sum m_i (W_i)_k = 0$.

The analogous definitions can be formulated for left superwords and left modules.

Unless otherwise stated, we consider the case of algebras with a finite alphabet $\mathcal{A}$. In this case the set of superwords (left, right, bilateral) over $\mathcal{A}$ constitutes a compact space $\mathcal{A}^\infty$ in the Tihonov product topology, which is induced by the discrete topology in $\mathcal{A}$. Right superwords, in respect to the left lexicographic ordering (and left superwords, in respect to the right lexicographic ordering) constitute a linear ordered set and each subset in this set has an infimum and a supremum.

By the lemma about the existence of a superword, such that each subword in it is a subword of a word in the given family of finite words of unbounded length, we have the following statement.

Proposition 1.7 *a) In each finitely generated nonnilpotent algebra A there exist nonzero superwords.*

b) Let M be a f.g. right A-module and A be a f.g. algebra. Then, if $\forall k \;\; MA^k \neq 0$, then there exist $m \in M$ and a right superword W, such that $mW \neq 0$. □

By the definition of a zero superword, we have that the set of zero superwords is open in Tihonov topology and the set of nonzero superwords is closed. After the existence of the infimum and the supremum for each set of right superwords, we have the following statement.

Proposition 1.8 *a) Let W be a superword, then the set of all right superwords, such that all their subwords are contained in W, has the maximal and the minimal elements.*

b) Let $mA^k \neq 0$, $\forall k$. Then the set of all right superwords W, such that $mW \neq 0$, has the maximal an the minimal superword.

c) If A is not nilpotent, then the set of all nonzero right superwords has the maximal and the minimal elements. □

Definition 1.9 a) A set of words $\{w_i\}$ is called distinguished, if the quotient algebra $A/\operatorname{Id}(\{w_i\})$ is nilpotent.

b) A set of superwords $\{W_i\}$ is called distinguished, if a set of words $\{w_i\}$ is distiguished, where each w_i is a finite subword in W_i.

Let u be a word in A, which is maximal in the set of all nonzero words of length $\leq n$. It is possible, that u cannot be extended to a word with greater length. Hence, to use the superword technique, we shall apply the following construction.

Construction 1. Let A be an algebra with generators $a_s \succ \cdots \succ a_1$. Let $a_1 \succ x$ and let us consider a free product $A' = A * F\langle x\rangle$.

Then each word u in A is a beginning of some superword in A'. If u is the maximal word in A in the set of all words of length $\leq |u|$, then the maximal superword in A', which begins with u, is a superword in A. If u is a superword in A and each its beginning subword has the above property, then u is a maximal superword in A'.

The following construction is useful for work with modules.

Construction 2. Let A be an algebra with generators $a_s \succ \cdots \succ a_1$, M be a f.g. right A-module with generators $m_k \succ \cdots \succ m_1$. Let $m_1 \succ a_s$, $a_1 \succ x$. Let $\tilde{A} = A \oplus M$, $\forall i,j \quad 0 = m_i m_j = a_i m_j$; $\mu\alpha \quad (\mu \in M, \alpha \in A)$ is the result of the action of α on μ, $\alpha\beta$ $(\alpha, \beta \in A)$ is defined as the product in A. Let us denote by A'' the factor of the free product $\tilde{A} * F\langle x\rangle/I$, where the ideal I is generated by elements xm_i.

In A'' a maximal right superword begins with m_k. Each word in $\tilde{A}$ can be extended to a superword in A''. If $MA^k \neq 0$, $\forall k$, then a maximal superword in $\tilde{A}$ begins with some m_i. If u is the maximal word in A in the set of all words with length $\leq |u|$ and with a nonzero action on m_i, then, after a suitable enumeration of m_i, a maximal superword in A'' is a superword in $\tilde{A}$. If u is a superword in $\tilde{A}$ and each its beginning has the above property, then u is a maximal superword in A''.

In the study of the nilpotency problem, the following statement is helpful.

Proposition 1.10 *If in an algebra (a semigroup) there is no nonzero periodic superword, then all its words are nilpotent.* $\square$

Let us note that the algebra A_W has this property, where W is a u.r. non-periodic word (see Definition 1.30).

1.5 The growth in words and algebras

The notion of the growth can be applied to measure the number of relations in an algebraic system and its "infinity" degree. It is, in particular, a generalization of the dimension notion, to the infinite dimensional case. The problem about the behaviour of the growth function in groups, in semigroups and in algebras was studied by many authors. Rather ordinary is the polynomial (in the commutative case, for example) and the exponental (in free algebras) growth. J.Milnor posed the problem about the existence of groups of an intermediate growth. This problem was solved by R.I.Grigorchuk [16]. The growth of a finitely defined monomial algebra is either polynomial, or exponental. V.A.Ufnarovski [49] constructed an example of a finitely defined algebra of an intermediate growth: it is the enveloping algebra of an infinite-dimensional Lie algebra. A.A.Kirillov and M.L.Koncevich [26] constructed a relatively free Lie algebra of an intermediate growth: the Lie algebra of generic vector fields on a variety. The problem about the behaviour of the growth function for algebras is, therefore, of the interest.

Definitions. Let A be an algebra with the fixed set of generators. By $\langle x\rangle$ will be denoted the ideal, generatedby the element x, by (α) will be denoted the ideal, generated by the set α. The growth function $V_A(n)$ of an algebra A is the dimension of the space, generated by all words with length $\le n$. $V_A(0)=0$, $V_A(1)$ is the number of generators. Let $T_A(n)=V_A(n)-V_A(n-1)$. If A is a homogeneous algebra and all A_i has the homogeneous degree 1, then $T_A(n)$ is the dimension of the space, generated by all words of length exactly n. The Gilbert series $H_A(x)$ is the generating function $\sum V_A(n)\cdot x^n$. It is easy to see that $\sum T_A(n)\cdot x^n=(1-x)\cdot H_A(x)$. The growth function of an infinite word W can be defined in an analogous way: $V_W(n)$ is the number of its different subwords of length $\le n$, $T_W(n)=V_W(n)-V_W(n-1)$ is the number of its different subwords of length exactly n. It can be proved that for each algebra with the fixed set of generators there exists a monomial algebra with the same growth function. In what follows, we shall consider only the monomial case. Then $T_A(n)$ is the number of nonzero words in A of length exactly n.

Let us note that the growth of a free k-generated algebra A_k and of a free group G_k is exponental: $T_{A_k}(n)=k^n$, $T_{G_k}(n)=k(k-1)^n$. If L_k is a free k-generated Lie algebra, then $T_{L_k}(n)$ is the number of correct words of length n (see subsection 2.3.1) and $T_{L_k}(n)>k^n/n$. On the other hand, the growth of solvable groups and of PI-algebras is polynomial.

The following theorem describes a possible "anomalous" behaviour of growth functions.

Theorem 1.11 *a) Let $\psi(n)=e^{o(n)}$, i.e., $\lim_{n\to\infty} ln\psi(n)/n=0$. Let $\varphi(n)=n(n+3)/2+1/o(1)$, i.e., $\lim_{n\to\infty}(\varphi(n)-n(n+3)/2)=\infty$. Then there exist an algebra A and two infinite subsets $\mathcal{K}$ and $\mathcal{L}$ in $\mathbb{N}$, such that*

1) $V_A(n)>\psi(n)$, for all $n\in\mathcal{K}$:

2) $V_A(n)<\varphi(n)$, for all $n\in\mathcal{L}$.

If $\psi(n)$ is a polynomial, then A can be chosen, as a PI-algebra.

b) Let $\psi(n)<Cn^k$. Then there exists a PI-algebra, which satisfies the conditions of a) for ψ and φ.

Let, for example, $\psi(n)=e^{\sqrt{n}}$, $\varphi(n)=n(n+3)/2+ln(n)$. Then the growth function of A sometimes has a slow growth and lags behind φ, then begins to grow very fast and outstrips ψ, and all this occurs infinitely many times.

Proof. We have to consider two auxiliary algebras.

1. The algebra $A^\infty=\Phi\langle a,b\rangle/(b)^2$. Words in this algebra are those words, which don't have more then one occurence of the letter b. Then $T_{A^\infty}(n)=n+1$, hence $V_{A^\infty}(n)=n(n+3)/2$. Obviously, we have the following proposition.

Proposition 1.12 *For each constant C there exists $K(C,\varphi)$, such that $V_{A^\infty}(n) + C < \varphi(n)$, for $n > K(C,\varphi)$.* $\square$

2. The algebra $A^n = \Phi\langle a,b\rangle / I_b^n$, where I_b^n is the ideal, such that its words don't contain subwords of the form ba^kb, $k < n$. Obviously, $T_{A^n}(k) = T_{A^\infty}(k) = k+1$, if $k < n$. Also, obviously, $T_{A^n}(k) \geq 2^{[k/n]}$. To prove this, it is enough to consider the set of words, such that in positions with the number, non-divisible by n, they have a. The number of such words is $2^{[k/n]}$. Therefore, the following statement is valid.

Proposition 1.13 *There exists $N_\psi(n)$, such that, if $k > N_\psi(n)$, then the following inequalities hold*

$$V_{A^n}(k) > T_{A^n}(k) \geq \psi(k).$$

$\square$

Hence, the algebra A^n has an exponental growth. The set of words of the algebra, under the construction, is similar to the same set of A^∞, for words of some length, and is similar to the same set of the algebra A^{l_i}, for other length.

The main construction. Let $\alpha : l_1 < k_1 < l_2 < k_2 < \ldots < l_i < k_i < l_{i+1} < \ldots$ be a sequence of positive integers. By $I(\alpha)$ will be denoted the ideal generated by the following sets $\mathcal{W}_\infty$ and $\mathcal{W}_\in$:

$$\mathcal{W}_1 = \{W \mid l_i < |W| < k_i \quad \& \quad \exists k < l_i : \; ba^kb \subset W\},$$
$$\mathcal{W}_2 = \{W \mid k_i < |W| < l_{i+1} \quad \& \quad \exists k < l_{i+1} : \; ba^kb \subset W\}.$$

Let $W(\alpha) = \mathrm{Wd}(a,b) \backslash I(\alpha)$, $A(\alpha) = \Phi(a,b)/I(\alpha)$.

Obviously, if $l_i \leq n < k_i$, then $T_{A(\alpha)}(n) = T_{A^{l_i}}(n)$, and, if $k_i \leq n < l_{i+1}$, then $T_{A(\alpha)}(n) = T_{A^\infty}(n) = n+1$. Proposition 1.13 allows to choose k_i so, that the inequality $V_{A(\alpha)}(k_i - 1) > T_{A(\alpha)}(k_i - 1) = T_{A^{l_i}}(k_i - 1) > \psi(k_i - 1)$ holds, and Proposition 1.12 allows to choose l_{i+1} so, that the inequality $V_{A(\alpha)}(l_{i+1} - 1) < \varphi(l_{i+1} - 1)$ holds. In this case the algebra $A(\alpha)$ satisfies the theorem's conditions. The first part of the theorem is proved.

The proof of b) is analogous. It is enough to mention that the algebra $F\langle a,b\rangle / \mathrm{id}(b)^k$ is a PI-algebra and the number of words with k occurences of the letter b, such that the distance between each two occurences of b is greater than R, has the n^k growth. $\square$

Remark. On the set of all growth functions can be defined the following relations: the relation of the nonstrict order: $f \succ g$, if for some c and k $cf(kn) \geq g(n)$, for all n; the equivalency relation: $f \equiv g$, if $f \succ g$ and $g \succ f$. The equivalency class of a growth function of an algebra (or a group, a semigroup, a Lie algebra) doesn't depend on the generators choice.

In the case of Lie algebras the following theorem holds.

Theorem 1.14 *Let $\psi(n) = e^{o(n)}$, i.e., $\lim_{n\to\infty} ln(\psi(n))/n = 0$. Let $\varphi(n) = n + o(1)/n$, i.e., $\lim_{n\to\infty}(\varphi(n) - n) = \infty$. Then there exist a Lie algebra L and two infinite subsets $\mathcal{K}$ and $\mathcal{L}$ of positive integers, such that*

a) $V_L(n) > \psi(n)$, for all $n \in \mathcal{K}$;

b) $V_L(n) < \varphi(n)$, for all $n \in \mathcal{L}$.

Let, for example, $\psi(n) = e^{\sqrt{n}}$, $\varphi(n) = n + ln(n)$. Then there exists an algebra with the growth function, which is sometimes smaller, than φ, and sometimes is greater, than ψ.

The idea of proof. To each algebra A with the fixed set of generators we can correspond the algebra $A^{(-)}$, which is a subalgebra in A^- and is generated by generators of A. Obviously, $T^{(-)}_{A^\infty}(k) = 1$, if $k > 1$, and that $T^{(-)}_{A^n}(k)$ has an exponental growth. Therefore, a Lie algebra L, which satisfies the theorem conditions, can be constructed as an algebra of the type $A(\alpha)^{(-)}$. □

We can formulate the following statement about the number of subwords in an infinite word.

Theorem 1.15 *Let $\psi = e^{o(n)}$, $\varphi(n) = n + 1/o(n)$. Then there exists an infinite word W and two infinite sets of positive integers $\mathcal{K}$ and $\mathcal{L}$, such that*

a) $T_W(n) > \psi(n)$, for all $n \in \mathcal{K}$ (by $T_W(n)$ is denoted the number of different subwords in W of the length exactly n);

b) $V_W(n) < \varphi(n)$, for all $n \in \mathcal{L}$.

Moreover, we can take W as an u.r. word.

(The definition of a uniformly recurrent word see further in 1.6.)

As each subword in W of length n can be extended to a subword of length $n+1$, then $T_W(n+1) \geq T_W(n)$. In the case of the equality, each part of length n in W uniquely defines the following symbol, hence, W is periodic. In this case $T_W(n) = const = k$, where k is the period length and $n > k$. Therefore, the condition on φ cannot be improved.

The idea of proof. We shall construct W step by step. On the k-th step of the first kind we shall construct words $a_k = b^l_{k-1} a_{k-1} b^l_{k-1}$ and $b_k = b^{l+1}_{k-1} a_{k-1} b^{l+1}_{k-1}$. On the k-th step of the second kind we shall construct words $a_k = u_k(a_{k-1}, b_{k-1})$ and $b_k = v_k(a_{k-1}, b_{k-1})$, where words u_k and v_k are different, are not powers, contain all subwords of length m_k and the length m_k on each step is sufficiently big. The word a_k contains a_{k-1}, therefore all words a_k can be united in the infinite word W.

The following statement holds.

Proposition 1.16 *a) The number of subwords of length n in the word $b_{k-1}^{\infty}a_{k-1}b_{k-1}^{\infty}$ is not greater, than $n+|b_{k-1}|+|a_{k-1}|$, i.e., has the growth $n+const$.*

b) The number of subwords of length n in an infinite word, which contains all subwords from a_{k-1} and b_{k-1}, is not less, than $2^{n/\max(|a_{k-1}|,b_{k-1}|)}$. □

By this proposition it follows that, if l is sufficienly big, then the number of subwords of length n in $a_k = b_{k-1}^l a_{k-1} b_{k-1}^l$ and in $b_k = b_{k-1}^{l+1} a_{k-1} b_{k-1}^{l+1}$ and also in their products has the growth $n + const$ in big intervals.

On the other hand, if for the second kind step we choose m_k sufficiently big, then the number of subwords in $a_k = u_k(a_{k-1}, b_{k-1})$ has an exponental growth in big intervals.

Therefore, using the reasoning, similar to the same in the proof of Theorem 1.11, we, by constructing on each step substitutions of the first and the second kind, shall obtain the required word. It will be uniformly recurrent, because each word, constructed by this system of substitutions, when a_{k-1} and b_{k-1} occur in a_k and b_k, will be uniformly recurrent.

On the other hand, the conditions of these theorems cannot be improved. Indeed, both for functions T_A, V_A and for T_W, V_W the following statement holds.

Theorem 1.17 *There exist the following limits:*

a) $\lambda = \lim_{n\to\infty} ln(T(n))/n$;

b) $lim_{n\to\infty}(T(n)-n)$ and, therefore, $lim_{n\to\infty}(V(n)-n(n+3)/2)$;

c) if L is a Lie algebra, then there exist the limit $lim_{n\to\infty}(V_L(n)-n)$.

Let us prove at first the case c). If $V_L(n+1) = V_L(n)$, then the algebra L is finite dimensional and the above limit is $-\infty$. The same is true for associative algebras also (if $V_G(n+1) = V_G(n)$, then the semigroup G is finite). Hence, in the infinite case, $T(n) \geq 1$.

As for each superword W there exists a monomial algebra with the same growth function, then the general case can be reduced to the monomial one. The case b) of Theorem 1.17 will be proved in 2.4. Let us prove the case a). If $m + k \geq n$, then the beginning part of length m and the end part of lenght k uniquely define the word. Therefore, we have the following proposition.

Proposition 1.18 *If $m+k \geq n$, then $T(m)\cdot T(k) \geq T(n)$.* □

Let $\tau(n) = ln(t(n))$. Then $\tau(k) \leq \tau(m) + \tau(n)$, if $m + n \geq k$. So, it remains to use the following well known analytic fact.

Proposition 1.19 *If $\tau(n) \geq 0$ and $\tau(m+n) \leq \tau(m)+\tau(n)$, then there exists the limit* $\lim_{n\to\infty} \tau(n)/n$. □

Remark. The condition of Theorem 1.17 a) means that the infimum in the conditions of Theorems 1.11 and 1.15 cannot be increased. The conditions of Theorem 1.17 b) and c) mean that the supremum in the conditiones of Theorems 1.11, 1.14 and 1.15 cannot be diminished. So, can be stated the problem about the validity of a) for Lie algebras. We suppose that the answer here is negative. The technique of the work [49] allows to prove that, if $\lim_{n\to\infty} \ln(V_L(n))/n = 0$, then $\lim_{n\to\infty} \ln(V_{U(L)}(n))/n = 0$. But, if only the infimum equals zero $\underline{\lim}_{n\to\infty} \ln(V_L(n))/n = 0$, then this technique doesn't work. So, the problem can be formulated in the following way: let L be a Lie algebra and $\underline{\lim}_{n\to\infty} \ln(V_L(n))/n = 0$, is that true that $\lim_{n\to\infty} \ln(V_L(n))/n = 0$?

Theorem 1.17 is a motivation of the following definition.

Definition 1.20 An algebra (a semigroup) A has a slow growth, if $T_A(n) \leq C$ (hence, $V_A(n) = O(n)$).

Theorem 1.21 *Let $d = \overline{\lim} T(n)$, $e = \underline{\lim} T(n)$. Then $e^2 \geq d$.* □

If A has a relation or contains a word u, which is not a subword in W, then λ is strictly less, than k – the number of letters in the alphabet.

We know that each u.r. word is a word, which is minimal in respect to the set of its subwords (Theorem 1.17). But, how big can be this subset? The answer is unexpected: almost as big as possible. The following statements hold.

Theorem 1.22 *For each $\lambda < k$ there exists a u.r. word W, such that $T_W(n) > \lambda^n$, for all n.*

The idea of proof. Let us use the following argument (which is related to the idea of the Golod-Shafarevich counterexample): the long word prohibition can have as small influence on λ, as possible. We shall construct the u.r. word W step by step. We allow u to be a subword in W, but prohibite all sufficiently long words without the u occurence. But such words constitute a negligible set, if length of these words is sufficiently big. On each step we make the permission of the word of the minimal length and the, related to it, prohibition, and after this we come to another word. The proof uses the techique of Ufnarovski graphs: let us consider the graph, which verteces are the permited words with the length smaller, than the maximal length of the prohibition. Verteces u and v are connected with the arrow, marked by a_i, if $ua_i = a_jv$. In the beginning the graph is connected and the connectivity on each step is preserved by construction. The set of sufficiently long paths, which don't contain the vertex u, becomes negligible by its relative cardinality and by its influence, so we can prohibite them. And we get new relations and new graph.

1.5.1 The Gelfand-Kirillov dimension. The superdimension

Definition 1.23 If the following limit exists, then its value is called the Gelfand-Kirillov dimension of an algebra A and it is denoted by $\mathrm{GKdim}(A)$ or $\mathrm{GK}\,(A)$:

$$\mathrm{GKdim}(A) = \lim_{n\to\infty} \ln(V_A(n))/\ln(n).$$

It means that $V_A(n) \sim n^{\mathrm{GKdim}(A)}$. $\mathrm{GKdim}(A)$ can be zero (then $\dim(A) < \infty$), can be 1, can be any number ≥ 2 and can be ∞.

The Gelfand-Kirillov dimension doesn't depend on the generators choice. If A is finite-dimensional over its center (in particular, commutative), then $\mathrm{GKdim}(A)$ equals to the transcendence degree of the center. A.V.Grishin proved the same statement for $M(n,s)$ – the s-generated algebra of generic matrices. He proved that $\mathrm{GKdim}(M(n,s)) = n^2 + ns$ (see [18]). The Gelfand-Kirillov dimension of a PI-algebra is not greater, than its Shirshov height and, hence, is bounded. The contrary is also true. In the representable case $\mathrm{GKdim}(A)$ equals to the essential height (see further). I.M.Gelfand and A.A.Kirillov introduced this dimension in the work [11] and proved there that the enveloping algebra $U(L)$ of an n-dimensional Lie algebra L has dimension n.

In [49] V.A.Ufnarovski introduced the notion of the superdimension.

Definition 1.24 If the following limit exists, then its value is called the superdimension of an algebra A and is denoted by $\mathrm{DIM}(A)$:

$$\mathrm{DIM}(A) = \lim_{n\to\infty} \ln(\ln(V_A(n)))/\ln(n)$$

It means that $V_A(n) \sim e^{n^{\mathrm{DIM}(A)}}$.

$\mathrm{DIM}(A)$ doesn't depend on the choice of A generators and can take values from the segment $[0,1]$. If A is free, then $\mathrm{DIM}(A) = 1$, if A has a polynomial growth, then $\mathrm{DIM}(A) = 0$. V.A.Ufnarovski proved the following theorem [49].

Theorem 1.25 *Let L be an infinite dimensional Lie algebra and $\mathrm{DIM}(L)$ exists. Let us denote by $U(L)$ its enveloping algebra. Then $\mathrm{DIM}(U(L)) = (1 + \mathrm{DIM}(L))/2$.*

Remark. Usually the superdimension is defined, as the supremum

$$\mathrm{DIM}(A) = \overline{\lim}(\ln\ln V_A(n)/\ln n).$$

If we take the infimum instead of the supremum, then we get the definition of $\underline{\mathrm{DIM}}(A)$. For infinite dimensional Lie algebras V.A.Ufnarovski proved the inequality $\underline{\mathrm{DIM}}(U(L)) \geq (1 + \underline{\mathrm{DIM}}(L))/2$. V.M.Petrogradski noted [41] that this inequality may be strict.

V.A.Ufnarovski considered the following example. Let L_1 be an infinite dimensional Lie algebra with the base $\{e_i, i = 1, 2, \ldots\}$ and the product $[e_i, e_j] = (i-j)e_{i+j}$. Obviously, L_1 is finitely defined. For this algebra

$$\mathrm{DIM}\, L_1 = 0, \qquad \mathrm{DIM}\, U(L_1) = 1/2.$$

Therefore, a finitely defined associative algebra $A = U(L_1)$ with an intermediate growth $V_A(n) \sim e^{\sqrt{n}}$ is constructed.

In the end of this subsection let us mention the V.N.Gerasimov result [12].

Theorem 1.26 *A graded algebra with the unique homogeneous defining relation has a rational Gilbert series.* □

1.6 Finite subwords and uniformly recurrent words. The compactness considerations

Notations and definitions. A segment or a subword v of a word W is called a set of symbols, which occurs in W in succession. The place in W, where this segment occurs is called the occurence of v in W. By $|W|$ will be denoted the set of all finite subwords of an infinite word W. The notation $W_1 \sqsupset W_2$ means that $|W_1| \supset |W_2|$ and $W_1 \sqsupseteq W_2$ means that $|W_1| \supseteq |W_2|$. In the first case, we say that W_1 is richer in segments, than W_2, in the second case, we say that $W_1$is not poorer in segments. Two words U and V are called equivalent $U \sim V$, if $|U| = |V|$. The notions "not richer in segments" and "poorer in segments" are obvious.

Definition 1.27 An (infinite) word U is a sequence $\{u_n\}_{n\in\mathbb{Z}}$ of symbols. The shift operator τ is the operator $U \to \tau(U)$, which is defined in the componentwise way: $\tau(\{u_n\}_{n\in\mathbb{Z}}) = \{v_n\}_{n\in\mathbb{Z}}$, $v_n = u_{n+1}$.

Definition 1.28 The distance between words W_1 and W_2 is the number $d(W_1, W_2) = \sum_{n\in\mathbb{Z}} \lambda_n 2^{-|n|}$, where $\lambda_n = 0$, if symbols in the n-th position of W_1 and W_2 coincide, and $\lambda_n = 1$, otherwise. (This distance is called the Hemming distance).

The following statements hold.

1. The set $\mathbf{W}$ of all words is a compact metric space.

2. The shift operators τ and τ^{-1} are continuous, i.e., $\tau : \mathbf{W} \to \mathbf{W}$ is a homeomorphism. Moreover

$$d(\tau(W_1), \tau(W_2)) \le 2d(W_1, W_2), \quad d(\tau^{-1}(W_1), \tau^{-1}(W_2)) \le 2d(W_1, W_2).$$

3. For each word W by $\widehat{W}$ will be denoted the set of all words, such that a shift of W is infinitely close to each of them, i.e., $U \in \widehat{W} \Leftrightarrow \forall \varepsilon > 0 \quad \exists n \in \mathbb{Z} : d(U, T^n W) < \varepsilon$, i.e., $\widehat{W}$ is the closure of the set $\{T^n W, n \in \mathbb{Z}\}$.

The following properties of a word U are equivalent:

a) $U \in \widehat{W}$,

b) U belongs to a closed orbit of W,

c) $|U| \subset |W|$ (i.e., $U \sqsubset W$).

Let $U < V$, if $\widehat{U} \subset \widehat{V}$, and let $U \vdash V$, if $\widehat{U} = \widehat{V}$.

Proposition 1.29 $U < V \Leftrightarrow |U| \subset |V|$ *(each finite segment of U occures in V).* □

Example. Let us consider the sequence $123\ldots$, which is obtained, when we write all positive integers in the decimal notation in succession. Each combination of numbers occurs in it, hence its closed orbit coincides with the set of all words.

Definition 1.30 A word U is called uniformly recurrent, if $\forall k \quad \exists n(k)$, such that each segment of length k in U is contained in each segment of length $n(k)$.

The equivalent definition is: $\forall u \subset U \quad \exists n(u) : \forall v \subset U \quad |v| \geq n(u) \Rightarrow u \subset v$.

Theorem 1.31 *The following two properties of an infinite word W are equivalent:*

a) for each k there exists $N(k)$, such that each segment in W of length k is contained in each segment of length $N(k)$;

b) if each finite segment of a word V is also a segment in W, then each finite segment in W is a segment in V.

Proof. Let us prove at first the simpler implication a)$\Rightarrow$b).

Let s be an arbitrary segment in W, $|s| = k$ be its length, and let each segment in V is also a segment in W. We have to prove, that s occurs in V. Let us consider an arbitrary segment C in V of length $N(k)$. Then C occurs in W, hence, s occurs in C. But C occurs in V, therefore s occurs in V.

Let us now prove the implication b)$\Rightarrow$ a). Let us suppose that a) doesn't hold. Then there exists a subword s in W and segments in W of arbitrary big length, which don't contain s. We shall prove that there exists a word V, such that each its segment is also a segment in W, but which doesn't contain s. This contradicts b). The above statement is a consequence of the following lemma, which will be also useful for us in what follows.

Lemma 1.32 (the compactness lemma) *Let $\mathcal{M}$ be a set of words of unbounded length over a finite alphabet $\mathcal{A}$. Then there exists an infinite word V, such that each its subword is also a subword of some word in $\mathcal{M}$.* □

Corollary 1.33 *If u is a subword in W an W contains subwords of arbitrary big length, which don't contain u, then there exists a word W', which is poorer in segments, than W. Moreover, we can choose W' so, that W' doesn't contain u.* □

Corollary 1.34 *Each descending, in respect to $\sqsupseteq$ relation, chain of superwords has its infimum.*

Proof. Let us correspond to each superword the set of its subwords of length n. All such sets, corresponding to words in the chain, are ordered by the inclusion. The set of all words of a given length is finite, hence, there exists a word w_n of length n, which is a subword in each word in the chain. Now we can apply Lemma 1.32 for words w_n. $\square$

Corollary 1.35 *For each word W there exists a word W', such that W' is poorer in segments, than W, $W \sqsupseteq W'$.*

Proof. It is enough to apply the Zorn lemma. $\square$

We have the following theorem, which states that we can construct an u.r. word from the segments of an arbitrary word.

Theorem 1.36 *Let W be an infinite word. Then there exists a u.r. word $\widehat{W}$, such that all its subwords are also subwords in W.* $\square$

This theorem is exceptionally important in the word combinatorial analysis, because it often allows to reduce the study of arbitrary words to the study of u.r. words.

1.6.1 Superwords and dynamics

Let us consider the action of the shift operator τ on the set of all superwords. An invariant subset is a subset in the set of all superwords, which is invariant in respect to τ action. A minimal closed invariant subset (m.c.i.s.) is a closed (in respect to the introduced above metric) invariant subset, which is nonempty and doesn't contain any closed invariant subsets, except empty set and itself. Usually a m.c.i.s. will be denoted by N.

Properties of closed invariant subsets. a) A m.c.i.s. N is a closed orbit of each of its elements.

b) Each two m.c.i.s either coincide, or have the empty intersection. In the last case the distance between them have the upper and the lower bound.

c) The following properties of W are equivalent: 1) W is u.r.; 2) $|W|$ is a minimal in respect to the inclusion in the class of sets of the form $|V|$ (for each U, if $W \sqsubseteq U$, then $U \sqsubseteq W$, i.e., $U \sim W$); 3) the closed orbit of W is minimal and is a m.c.i.s.

The property a) means the almost returning to any point x: N is a closed orbit of each of its points, i.e., the point $y = f(x)$ also. It means, that the orbit of y can be infinitely close to any point in N, i.e., to x also! However, a more strong and interesting statement holds: N has the property of the uniform almost returnability.

Theorem 1.37 *Let U be an open set, such that $U \cap M \neq \emptyset$. Then there exists $n = n(U) \in \mathbb{N}$, such that for all $y \in N$ an iteration $f^{(k)}(y) \in U$ for some k, $1 \leq k \leq n$.* $\square$

Theorem 1.38 *Let L be a closed invariant set. Then there exists a m.c.i.s. $N \subseteq L$.*

Proof. The intersection of each chain of closed nonempty sets is nonempty because of their compactness. The intersection of invariant sets in also invariant. Now we can use the Zorn lemma for a set of closed inveriant sets, oredered by the inclusion. $\square$

Remark. We obtained one more proof of the existence of an u.r. word, which is constructed from the given word U segments (Theorem 1.36): let N be a m.c.i.s., which is contained in the closed orbit $\overline{U}$ of the word U. Let $W \in N$, then W is a required u.r. word.

The notion of a u.r. word has a dynamical sense. If on the k-th position of the sequence W is 0 or 1, depending on the belonging $f^{(k)}(y_0) \in U$, then, in the generic case (when $f^{(k)}(y_0)$ doesn't belong to U border for all k) W is u.r. and each u.r. word can be obtained in the similar way. We can take the shift operator for f and for N a m.c.i.s. in the set of all superwords with the Tihonov topology.

We shall see (Theorem 2.175) that monomial algebras has either linear growth, or their growth is not less, than $n(n+3)/2$, and the equality case corresponds to the case, when $T(n) = n + 1$. All such algebras are defined by those words, which are described in the section c) of the following theorem.

A.T.Kolotov studied algebras of the slow growth. He constructed a semigroup, which satisfies the identity $x^3 = 0$, and the number of words of length n in it equals $n+1$. For the growth $< n+1$ such example is impossible: all words turn out to be weakly pseudoperiodic of the bounded order.

Theorem 1.39 *The following classes of sequences from 0 and 1 are almost equivalent in those sense, that there exist only countable number of sequences from one class, which don't belong to another:*

a) let $f(n) = [\alpha n + \beta]$, $\alpha, \beta \in [0,1]$. Elements of the sequence are defined by the equality $a_n = f(n+1) - f(n)$;

b) for each k and for each two segments of length k, the number of occurencies of 1 in each segment differs by not more, than 1;

c) for each k the number of different subwords of length k equals $k+1$ (it can be proved that it is the minimal possible number of subwords in a nonperiodic word).

All exceptional sequences are as follows: in the case a), when $\alpha \in \mathbb{Q}$; in the case c) it is the sequence $\ldots 111000 \ldots$ and all sequences, which can be produced from it by means of a finite number of substitutions of the type $1 \to 0^k 1$, $0 \to 0^{k+1} 1$ or of the type $0 \to 1^k 0$, $1 \to 1^{k+1} 0$.

A sketch of the proof. Let us consider a sequence, which satisfy the condition b) or c). For each its unit let us count the number of zeroes, which separate it from the next unit. We shall get either infinity (the special case), or the sequence of nonnegative integers. There are not more, than two different numbers in this sequence. If we substitute the smaller by 0 and the greater by 1, then we shall obtain a uniform sequence. The contrary is also true. □

A sequence from a) corresponds to the following dynamical system. Let us take a segment in the circle of length α. If the operator f is the rotation by the angle α, then the hit into this segment corresponds to 1 and the miss to 0.

Let us note that to the sequence from the A.T.Kolotov example corresponds $\alpha = (\sqrt{5}-1)/2$.

2 Apllications of the word analysis to the problems of Burnside type

Combinatorial effects, related to the periodicity, play an important role in the problems of Burnside type. The notions of almost periodicity and the uniform recurrency will be the base of our reasoning. An application of these notions allows to obtain simple proofs of such well known theorems in the ring theory, as Shirshov theorem about the boundedness of heights in a PI-algebra, Ufnarovski independency theorem, Razmyslov-Kemer-Braun theorem about the nilpotency of the radical in a finitely generated PI-algebra and so on.

2.1 The periodicity in words

The positive solution of problems of the Burnside type means the appearence of the periodicity. Here we study periodic sequences, which are the combinatorially most "pure" case.

2.1.1 The properties of the sequence u^∞

Let us define the operator δ, which acts on words. If a word u is a product $u = u'b$, where b is a letter, then let $\delta(u) = bu'$. Let $m = |u|$.

Proposition 2.1 *The set $\{\delta^k(u)\}$ is the set of words, which are cyclically conjugate to the word u. Let $u = v^q$, where the word v is not a power, then* $\delta^k(u) = u \Leftrightarrow q \quad \text{divides} \quad k$. □

The following proposition connects shifts of a word u^∞ and the cyclic conjugation operator δ. By u will be denoted a word, which is not a power.

Proposition 2.2 *a) Let $|v| = N|u| = |v'|$ and let the first letter in v' occurs in v on distance 1 to the right of the v first letter. Then $v' = \delta(v)$.*

b) If the first letter in v' occurs in v on distance k to the right of the v first letter, then $v' = \delta^k(v)$.

c) Each two subwords in u^∞ of length $N|u|$ are cyclically conjugate and they coincide, only when the distance between their first letter is divisible by the period.

Proof. As $|v| = N|u|$, then the distance between the last letter in v' and the first letter in v is divisible by $|u|$ and, hence, they coincide. Therefore, $v' = \delta(v)$. The property b) is a consequence of the property a), and the property c) is a consequnce of the property b). □

The following useful statement is a consequence of the above proposition.

Proposition 2.3 *a) The beginning subword of length m uniquely defines the word from A_{u^∞}. If beginning subwords of length $|u|$ in two subwords v and v' coincide (v and v' are subwords in the superword u^∞), then one of them is a subword in another. If $|c| \geq |u|$ and d_1 and d_2 are lexicographically comparable, then at least one of words cd_1, cd_2 is not a subword in u^∞.*

b) Positions of the occurences of a word v of length $\geq |u|$ in u^∞ differs by a period multiple.

c) Let $|v| \geq |u|$, $v^2 \subset u^\infty$, then v is cyclically conjugate to a power of u. Therefore, nonnilpotent words in A_{u^infty} are exactly those words, which are cyclically conjugate to words of the form u^k. □

A.A.Mihalev proved the following statement.

Proposition 2.4 *If the square of a regular word $v^2 \subset u^\infty$, then either v is a period (i.e., v is cyclically conjugate to u), or $|v^2| \leq |u|$.* □

In the general case this statement is wrong: let $v = aba$, $v^2 = aba^2ba$, $u = aba^2b$, $v^2 \subset u^\infty$, but $v^2 \not\subset u$.

Proposition 2.5 *If all subwords in a word W of length $|u|$ are also subwords in u^∞, i.e., are cyclically conjugate to u, then $W \subset u^\infty$. In other words, the algebra A_u^F is defined by the set of relations of length $\leq |u|$.* □

Lemma 2.6 (about deletions and addings) *Let $t = t_1vt_2 \subset u^\infty$ and v is cyclically conjugate to u. Then, for all $k \geq 0$, $t_1v^kt_2 \subset u^\infty$. In particular, $t_1t_2 \subset u^\infty$ and $t_1v^2t_2 \subset u^\infty$.* □

The periodicity of an infinite word means its invariancy in respect to a shift. In the one-side infinity case there appears a pre-period, in the finite case there appear effects, related to the cutting off. Exactly this is the kernel of a great number of combinatorial reasonings. Proofs of Shestakov hypothesis, of the independency theorem, of Shirshov height theorem, of the theorem about the coincidence of the nilradical and the Jacobson radical in a monomial algebra, are examples. Below are often used combinatorial lemmas of this type.

Proposition 2.7 *Let $uW = Ws$, then uW is a subword in u^∞ and $W = u^n r$, where r is a beginning segment in u.*

Proof. If W is a beginning segment in uW, then, for all k, $u^k W$ is a beginning segment in $u^{k+1}W$, hence, uW is a beginning of $u^{\infty/2}$. $\square$

Proposition 2.8 *a) Let W be an infinite to the right word, $W = vuW' = vW'$. Then W is periodic with the period u and with the pre-period v.*

b) Let W be a finite word, $W = vuW'$ and vW' is a beginning of W. Then $W = vu^k r$, where r is a beginning segment in u. In other words, if a words ut and vt are incomparable, then both of them are pseudoperiodic of order not greater, than $\max(|u|, |v|)$.

c) Let e_i be proper beginnings of a non-cyclic word u, and f_i are its proper ends. Then, if $f_1 u^2$ is comparable with $f_2 u^2$, then $f_1 = f_2$; if $u^2 e_1$ is an end of $u^2 e_2$, then $e_1 = e_2$. $\square$

Definitions. A word $W = u^k$, where $k > 1$, is called cyclic or periodic. If $W = u^k r$, where r is a beginning segment in u, then W is called quasiperiodic. If $W = vu^k r$, where r is a beginning segment in u, then W is called pseudoperiodic, v is called its pre-period and u is called its period. The order of W is the minimal possible value of $|v| + |u|$. If $W = vu^k$, then W is called a pre-periodic word. Let us note that, if a word is pre-periodic of order m, then there exists an automaton with m states, which can type this word, and the opposite statement is also valid.

The following proposition is useful in a realisation of the inductive descent method.

Proposition 2.9 *If a pseudoperiodic word is not quasiperiodic, then, after deleting its first letter, its order diminishes by* 1. $\square$

There is no sense in considering the left or the right quasiperiodicity, because quasiperiodic words of order m are exactly subwords in periodic words of the period m, and, if $W = u^k r$, where r is a beginning in u, then $W = sv^k$, where v is cyclically conjugate to u and s is an end in v. Then $v = \delta^{|s|}(u)$. In the case of the pseudoperidicity, the situation is different. Actually, we defined (and shall mainly use) the left pseudoperiodicity. It can be proved that, if $|W| \geq 2m$, then the quasiperiodicity of W is a consequence of its left and right pseudoperiodicity of orders, not greater, than m. However, we shall need even more weak "periodicity".

Definition 2.10 A word W is called weakly pseudoperiodic of order m, if it can be represented as $au^k rb$, where r is a beginning of u and $|a| + |b| + |u| \leq m$.

We shall prove further that all words in algebras of the slow growth are weakly pseudoperiodic of a bounded order. A word is weakly pseudoperiodic of order $m + 1$, if each its beginning segment is pseudoperiodic of order m.

Definition 2.11 A word is called m-proper, if each its beginning is pseudoperiodic of order $\leq m$ and the word itself doesn't have this property. An m-proper word is called minimal, if each its end is also pseudoperiodic of order $\leq m$.

Proposition 2.12 *Each m-proper word contains the unique minimal m-proper word, which is its end. If an m-proper word, contains another m-proper word, then both of them contain the common minimal m-proper word, which is an end of both of them.* □

Proposition 2.13 *Let l be the number of letters in the alphabet, $X(R)$ be the number of all m-proper words of length R, $X_k(m)$ be the number of all m-proper words, which don't contain the k-th power of a word of length $\leq m$ and $T(m, R)$ be the number of m-proper words of length $\leq m$. Then*
a) $X(m) < l^{m+1}m$, $T(m, R) < Rml^{m+1}$;
b) $X_k(m) < l^m(l-1)m \cdot km = l^m(l-1)m^2k$.

Proof. The beginning segment of length m can be chosen in l^m ways. There are m ways to divide it on a period and a pre-period. The last letter can be chosen in $l-1$ ways. At last, an m-proper word, which doesn't contain a k-th power, has a length $\leq mk$. □

Proposition 2.14 *a) A minimal m-proper word u has the length not less, than $m+1$, and not more, than $2m$.*
b) The number $Y(m)$ of such words is not more, than $l^m(l-1)m$.

Proof. By the inductive supposition and by Proposition 2.9, we can assume that the beginning of u or its end of length $|u|-1$ is quasiperiodic of order exactly m. It remains to use the overlappings lemma. The item b) is a direct consequence of a).

Remark. It will be interesting to obtain the exact number of m-proper and minimal m-proper words.

2.1.2 Overlappings of words

Let us study an arrangement of subwords inside of a word. We are mainly interested in overlappings.

Lemma 2.15 (on overlappings) *If a subword of length $m+n-1$ occurs in the both two periodic words of periods m and n, then they are composed of identical subwords.* □

Proposition 2.16 *Let us consider two periodic sequences of periods m and n, $m \geq n$. If there exists a word of length $2m$, which is the segment in the both of them, then these two sequences coincide.* □

The following proposition is a consequence of the overlappings lemma.

Proposition 2.17 *a) If two pseudoperiodic words of orders $\leq n$ and $\leq m$, respectively, have the same segment of length $m+n-1$, then the join of these two words is a pseudoperiodic word of order $\leq \max(m,n)$.*

b) Two m-proper subwords cannot have a common segment of length $2m$.

Proof. The item b) is a direct consequence of a). Using Proposition 2.9 and the inductive descent, we can assume that both these subwords are quasiperiodic. Now a) is a consequence of the ovelappings lemma.

An overlapping of minimal m-types

Proposition 2.18 *a) If two equal quasiperiodic subwords of order m have a common segment of length m, then their join is also quasiperiodic of the same order.*

b) If two equal pseudoperiodic subwords of order m have a common segment of length m, then their join is also pseudoperiodic of the same order.

c) Two equal m-proper subwords cannot have a common segment of length $\geq m+1$.

d) There cannot exist a triple of m-proper subwords with a common symbol.

□

We are interested in quasiperiodic segments of a word. An m-proper word u will be called an m-end, if each its beginning segment is quasiperiodic of order $\leq m$. If u is a minimal m-proper word with the same property, then u will be called a minimal m-end.

Proposition 2.19 *a) Two different m-ends cannot have a common segment of length $m+1$. In particular, one of these m-ends cannot contain another.*

b) The number of all minimal m-ends is not greater, than $l^m(l-1)$.

Proof. The item a) is a consequence of the above reasoning; an m-end is uniquely defined by the last symbol and the last period. □

The solution of problems of the Burnside type in associative algebras is based on the reduction of words to the piecewise periodic form. Therefore, we study properties of periodic sequences.

Definition 2.20 Let $Y = \{u_i\}$ be a set of words. The height of a set of words W in respect to Y is called the minimal h, such that each word $w \in W$ can be represented as a product $w = u_{i_1}^{k_1} u_{i_2}^{k_2} \dots u_{i_r}^{k_r}$, where $r \leq h$. An algebra A has the height h over Y, if A is linearly representable by a set of words, which has height h over Y. In this case Y is called the Shirshov base of the algebra A.

Shirshov height theorem. *Let A be a finitely generated PI-algebra of degree m and Y be a set of words of degree $\leq m$. Then A has a bounded height over Y.* $\square$

This theorem allows the reducing to the piecewise periodic form. The local finiteness of algebraic PI-algebras and, in particular, the local nilpotency of nil-algebras of a finite index is its direct consequence. Also the boundedness of Gelfand-Kirillov dimension is a consequence of the height theorem.

In connection with Shirshov theorem there can be raised the problem about the description of those sets Y, over which A has a bounded height. This problem is a particular case of the problem about bases in PI-algebras.

The following theorem holds.

Theorem 2.21 (A.Ya.Belov) *A set of words Y is a Shirshov base in an algebra A, only when for each word u of length not greater, than the complexity m of A, $m = \mathrm{PIdeg}(a)$, the set Y contains a word, which is cyclically conjugate to some power of u.*

Proof. Let us note that $(uv)^n = u(vu)^{n-1}v$ and, if Y contains two cyclically conjugate words v_1 and v_2, then one of them can be deleted from the base. Therefore, the sufficiency condition is a consequence of Theorem 2.121 (see below). Let us prove the necessity. Let $|u| \leq n$ and u is non-cyclic. By Theorem 5.18, $A_u \in \mathrm{Var}(A)$ and, hence, A_u is a quotient algebra of A. By the A_u nilpotency, the projection of Y must contain a nonnilpotent element. By Proposition 2.3, the set of nonnilpotent elements in A_u is the set of those words, which are cyclically conjugate to powers of u. Therefore, Y contains a word of the required type. $\square$

Remark. The boundedness of the height of a PI-algebra over the set of words with degrees not greater, than the algebra complexity, was announced also by G.P.Chekanu [58].

We shall be interested also in estimations on the height h. The idea here is that, if the height is big, than a word can be linearly represented by smaller words.

Definition 2.22 A word W is called m-divided, if it can be divided on m lexicographically decreasing segments

$$W = w_1 w_2 \dots w_m, \quad \text{where} \quad w_1 \succ w_2 \succ \dots \succ w_m.$$

The following proposition explains the importance of the m-divisibility notion.

Proposition 2.23 (A.I.Shirshov) *a) Let a word W be m-divided, then each word, which can be produced from W by a nonidentical permutation of w_i, is lexicographically smaller, than W.*

b) If the following identity of degree m

$$x_1 \dots x_m = \sum_{\sigma \neq \mathrm{id}} \lambda_\sigma x_{\sigma(1)} \dots x_{\sigma(m)}$$

holds in an algebra A, then the word W is representable as a linear combination of lexicographically smaller words. $\square$

Therefore, a word, which cannot be represented as a linear combination of smaller words in a PI-algebra of degree m, cannot be m-divided.

Remark. If $t = t_1 \dots t_n$, then the condition $t \succ t_\sigma$, $\forall \sigma \neq \mathrm{id}$, is equivalent to the condition $t_1 \triangleright \cdots \triangleright t_n$ (where $\triangleright$ is the Ufnarovski ordering, see 2.3).

In the work [80], in connection with the height theorem, there was introduced the notion of a permutable semigroup: for each product $t = t_1 \dots t_n$, the equality $t = t_\sigma$ holds for some $\sigma \in S_n \backslash \mathrm{id}$ (let us note that the permutation σ depends on the set $\{t_i\}$). The weak permutability means that $t_\sigma = t_\tau$, for some n and $\sigma \neq \tau$. Also in this work was introduced the notion of an ω-permutable semigroup: for each infinite product $t = t_1 \dots t_n \dots$, the equality $t_1 \dots t_n = t_{\sigma(1)} \dots t_{\sigma(n)}$ holds for some n and $\sigma \in S_n \backslash \mathrm{id}$. Analogously can be defined the weak ω-permutability and the ω-permutability for bilateral products. In the same work it was proved that a non-diminishable u.r. word in a ω-permutable semigroup is periodic. If σ is independent from $\{t_i\}$, then the height is bounded over generators (see 2.1.3).

Another (equivalent) definition of the m-divisibility A word W is called m-divided, if it is of the form $s_0 v_1 s_1 v_2 \dots s_{m-1} v_m s_m$, where $v_1 \succ v_2 \succ \cdots \succ v_m$.

Proposition 2.24 *If a word W is of a form $W = s_0 v s_1 v \dots s_{m-1} v s_m$ and v contains m pairwise lexicographically comparable words (possible, with overlappings), then W is m-divided.*

Proof. Let us choose the highest subword in the v first occurence, the second highest in the v second occurence and so on. I.e., we found in W a decreasing chain of non-overlapping subwords. $\square$

Therefore, a word, which contains n lexicographically comparable subwords can be n-encountered only in an n-divided word.

Corollary 2.25 *If a word is not m-divided, then it cannot contain m non-overlapping subwords of the same m-type and, also, of the minimal m-type.* $\square$

Definition 2.26 By the m-type of a word W will be called its m-proper beginning and in the case, when W is pseudoperiodic of order m, by the m-type of W will be called W itself. The minimal m-type is the minimal m-proper subword in a proper m-beginning.

Our problem now is to find m pairwise lexicographically comparable subwords in the given word. The following lemma is exceptionally important. It establishes the alternative the pseudoperiodicity – the lexicographic comparability. This alternative is the base of the height estimations obtaining, of the independency theorem proof, of the proof of the theorem about the coincidence of the nilradical and Jacobson radical in a monomial algebra and so on.

Lemma 2.27 *a) Let $u_0 = W$, $W = v_1u_1 = v_2u_2 = \cdots = v_mu_m$, $|v_i| = i$. Then one of the following two statements holds:*

1) the words u_i are lexicographically comparable and constitute a chain;

2) the word W is pseudoperiodic with the pre-period v_i for some i and with the period s, $v_is = v_j$; the order of W is $\leq m$.

b) Let u be a noncyclic word, $u^{(i)}$ be its proper ends. Then the words $u^{(i)}uu$ constitute a chain.

Proof. The item a) is a direct consequence of previous proposition. The item b) is a consequence of the following fact: if a quasiperiodic word contains the square of a subword, which length is greater, than the period, then this word is a power of the period. □

Corollary 2.28 *a) Let $|v_i| \leq m$, for all i, and $v_i \neq v_j$, if $i \neq j$. Then, either the set of words $\{v_it\}$ is linear ordered in respect to the lexicographic ordering, or t is quasiperiodic of order $\leq m$.*

b) Let us define $(zv)^{(i)}$ with the help of the equality $(zv)_i(zv^)(i) = zv$. Let $|zv| > R > m$, then either $(zv)_R$ is pseudoperiodic of order $\leq m$, or words $((zv)^{(i)})_R$ constitute a linearly ordered set. □

Let us note that W can be a right superword also. If it is not pre-periodic, then all its ends are pairwise different. This case will be considered in the proof of the independency theorem.

The pseudo- and the quasiperiodicity naturally appears in problems of the Burnside type. The following lemma is a direct consequence of the above propositions.

Lemma 2.29 *a) Let $|z| = R$ and let z be pseudoperiodic of order $\leq m$. Then z contains the $[R/m]$-th power of a word of length $\leq m$.*

b) In the conditions of the previous lemma, let W be infinite, $\mathcal{K}$ be the set of indeces i, such that $(u)^{(i)} = V$, where V is an end of the superword W, and k_0 be the minimal number in $\mathcal{K}$, $v = v_{k_0}$. Then there exists a word s, such that $v_k = vs^{n_k}$, for all $k \in \mathcal{K}$, where $n_k = (k - k_0)/|s|$. □

Corollary 2.30 *If $|t| \geq mk$, then t contains either k-th power of a word with length $\leq m$, or m subwords, which constitute a chain.* □

This subwords can overlap, but, if t itself has m nonoverlapping occurences in W, then W is m-divided.

The following statement is a consequence of Proposition 2.13.

Proposition 2.31 *a) The number of m-types of words, which don't contain the k-th power of a word with length $\leq m$, is not greater, than $l^{m+1}m^2k$.*

b) Each word of length $\geq mk$ is either pseudoperiodic of order m and contains the k-th power of a word of length $\leq m$, or doesn't coincide with its m-type.

c) If W has the length greater, than $l^{m+1}m^4k^2$, then W has $l^{m+1}m^3k$ segments of length mk and m of them have the same m-type. □

Corollary 2.32 *If a word W has the length greater, than $l^{m+1}m^4k^2$, then either W is m-divided, or some word from $n < m$ letters repeats k times in succession.* □

Let us note that, if we don't use m-types, then we get an estimation of order l^{mk}: a word of the length m^2kl^{mk} contains m nonoverlapping occurences of a word of length mk.

Corollary 2.33 *Let A be an l-generated PI-algebra of degree m. Let all words from generators of length $\leq m$ are nilpotent of index k. Then A is nilpotent of index $l^{m+1}m^4k^2$.* □

The above estimations can be improved, if we shall use the following considerations: at first to establish the m-divisibility, it is enough to prove the repetition of the minimal m-type. Then, there are no triple overlappings of the same m-types, hence, there is no necessaty in the cutting the word into segments of length mk and in the establishing the repetition of m-types of segments. Ovellappings of the same type can be only double, and the number of segments is in mk times smaller. Let us give improved estimations.

Proposition 2.34 *a) If the length of a word W is greater, than $2l^{m+1}m^2k$, then either W is m-divided, or some word from $n < m$ letter repeats k times in succession. In the above corollary, the estimation $l^{m+1}m^4k^2$ can be substituted for $2l^{m+1}m^2k$.*

b) A non-m-divided word, contains less, than $2ml^{m+1}$ subwords, such that they are m-ends.

c) A non-m-divided word can be divided in ml^{m+1} pseudiperiodic segments of order $\leq m$. □

Therefore, a non-m-divided word c has the following form

$$c = c_0\omega_0c_1\omega_1\ldots\omega_{r-1}c_r,$$

where ω_i are quasiperiodic words of order $\leq m$ and of length $\geq 2m$ each, each c_i doesn't contain such subwords and the product of ω_i on the first letter of c_i is not pseudoperiodic of order m. Hence, $r < ml^{m+1}$ and $\sum |c_i| < ml^{m+1}$.

Let us reperesent ω_i as $\omega_i = u_i^{k_i} S_i$, where S_i is a beginning of ω_i and $|u_i| \le m$. We have the following theorem.

Shirshov height theorem. *Let A be a l-generated PI-algebra of degree m. Then A has a bounded, by the function $H(m,l)$, height over the set of words of degree $\le m$, where*

$$H(m,l) < 3ml^{m+1}$$

□

Remark. A thorough study of ω_i ends allows to improve the estimation: $H(m,l) < 2ml^{m+1}$.

We shall be interested in the appearence of the q-th power of a word, which contains the given letter. The following proposition is a consequence of Shirshov theorem.

Proposition 2.35 *Let A be a l-generated PI-algebra of degree m. Let the letter z has $(q+3)ml^{m+1}$ occurences in a word W. Then W is linearly representable by words, such that each of them contains a subword of the form $(zv)^q$.* □

This proposition will be useful in the obtaining estimations for the height with respect to the set of words, which degree is not greater, than the complexity.

2.1.3 The height theorem for semigroups

Ring identities allow to reduce words to the piecewise periodic form, but the period can be arbitrary big. In the semigroup case the situation is simpler. The following proposition holds.

Proposition 2.36 *Let S be a l generated semigroup with the identity $x_1 \dots x_n = x_{\sigma(1)} \dots x_{\sigma(n)}$ ($\sigma \neq \mathrm{id}$). Then S has a bounded height over the set of generators.*

Proof. As a lexicographically nondiminishable word is non-n-divided, then the set of such words has a bounded height over the set of words of degree $\le n$. Hence, it is enough to prove that, if $|u| > 1$, then the n-th power of u equals to a smaller word. So, our proposition is a consequence of the following statement.

Lemma 2.37 *Let i be the minimal number, such that $\sigma(i) \neq i$, then $\sigma(i) > i$. Let a word $t = b_1 a_1 \dots b_n a_n$ and $\forall i \forall j \quad b_j \succ a_i$. If $t_1 = b_1 a_1, \dots, t_{i-1} = a_{i-1} b_i$, $\quad t_i = b_i a_i b_{i+1}$, $\quad t_{i+1} = a_{i+1} b_{i+2}, \dots$, $t_{n-1} = a_{n-1} b_{n-1}$, $\quad t_n = a_n$, $t_\sigma = t_{\sigma(1)} \dots t_{\sigma(n)}$, then $t \succ t_\sigma$.*

Proof. The word t is of the form $ct_i e = cb_i a_i b_{i+1} e_1$ and the word t_σ is of the form $ct_{\sigma(i)} f = ca_{\sigma(i)} f_1$. As $b_i \succ a_{\sigma(i)}$, then $t \succ t_\sigma$. □

Remark. We can obtain an estimation $h \leq l(n-1)+1$ for the height h (an arrangenent of powers of $l-1$ generator has not more, than $n-1$ connected components, and an arrangement of powers of the last generator has not more, than n components).

Therefore, it is much easier to find a diminishable by σ decomposition of t for one permutation σ. It explains the height boundedness over generators for semigroups with polylinear identities. The existence of periods of length > 1 is a ring phenomenon, related to the necessaty to diminish a decomposition $t = t_1 \dots t_n$ with respect to actions of several permutations (the identity contains several terms). The reasoning, analogous to the same in the proof of Proposition 2.36 and Lemma 2.37, allows to establish the height boundedness over the set of words of degree $\leq k$ for an algebra with a polylinear identity, which contains not more, than $k+1$ nonzero terms (or, which is the same, with an identity of the form $x_1 \dots x_n = \sum_{\sigma \neq \mathrm{id}} a_\sigma x_{\sigma(1)} \dots x_{\sigma(n)}$, where not more, than k numbers a_σ, are nonequal to zero). As each relatively free algebra of complexity n has an unbounded height over the set of words of degree $n-1$ (see Theorem 2.21), then each polylinear identity in the algebra of $n \times n$ matrices has not less, than $n+1$ nonzero coefficients. In the case of complexity 2 this result can be improved. It is known that two 2×2 generic matrices (over an infinite field F) generate an absolutely free semigroup. Therefore, in the algebra of 2×2 matrices over an infinite field F there are no semigroup identities. (If F is finite, then the identity $x^n = x^m$ holds for some m and n).

It is possible to state the following problems.

Problems. Is it true that the number of nonzero terms in a (not necessary polylinear) identity in the algebra of $n \times n$ matrices converges to infinity, when $n \to \infty$? Is it true that this number is $\geq n+1$? On the other hand, in $\mathbb{M}_{2n}$ the standart identity st_{2n} holds, hence, $2n$ is a upper estimation for the minimal number of nonzero terms. Is this estimation exact? In the polylinear case it probably can be proved, using the algebra A_u (probably it can be proved that an identity with less, than $2n$ terms, destroys the period n). We can take a word from n different letters as u.

2.1.4 The periodicity in words and the independency theorem

The following theorem is a consequence of Lemma 2.27.

Theorem 2.38 (on superwords independency) *Let W be a minimal nonzero superword. Then one of the following two conditions holds:*

a) words $(W)_1, \dots, (W)_n$ are linearly independent;

b) W is pseudoperiodic of order $\leq n$, its pre-period is some $(W)_i$ and its period is an end of $(W)_n$.

Proof. Let us define infinite words $(W)^i$ by the equality $W = (W)_i(W)^i$. If b) doesn't hold, then the set $\{(W)^i\}$ constitutes a chain. Let $\sum \alpha_i(W)_i = 0$ and let $(W)^j$ be a minimal word in the set $\{(W)^i : \alpha_i \neq 0\}$. Then $W = (W)_i(W)^i \succ (W)_i(W)^j$, for $i \neq j$ and $\alpha_i \neq 0$. As $(W)_j$ is representable by a linear combination of $(W)_i$, then $W = (W)_j(W)^j$ is also representable by a linear combination of smaller words. But this contradicts the minimality of W.

□

Remark. a) The minimality can be substituted by the maximality in the above theorem formulation.

b) If all subwords in W of length $\leq n$ are nilpotent, then the words $(W)_1, \ldots, (W)_n$ are linearly independent.

Corollary 2.39 *a) Let A be a subalgebra in the algebra of $n \times n$ matrices and W be a minimal nonzero (super)word. Then W is pseudoperiodic of order $\leq n$.*

b) Let M be a f.g. A-module and $MA^k \neq 0$, for all k. Then the minimal superword in the algebra A'' (see the construction 2 in the subsection 1.4) is pseudoperiodic of order $\leq n+1$ (1 is added, because a generator of M is situated before a word in A). □

By the independency theorem for superwords and the construction 1 from 1.4, we have

Corollary 2.40 (the independency theorem by V.A.Ufnarovski) *Let*

(1) a word $W = a_{i_1} \ldots a_{i_n}$ is minimal with respect to the left lexicographic ordering in the set of all nonzero products of length $\leq n$;

(2) ends of W are nilpotent.

Then beginnings of W are linearly independent. □

This theorem was proved by V.A.Ufnarovski [52] and also by G.P.Chekanu [57].

Corollary 2.41 (Shestakov hypothesis) *a) Let A be a subalgebra in the matrix algebra $\mathbb{M}_n$, which has the fixed set of generators. Then, if all words of degree $\leq n$ are nilpotent, then A is nilpotent.*

b) Let A be an algebra with complexity n and with the fixed set of generators. Then, if all words of degree $\leq n$ are nilpotent, then A is locally nilpotent.

Proof. a) A has an exact representations by operators in an n-dimensional space. Let $m_1, \ldots, m_n$ be a base in this space. Now it remains to apply the construction 2 and the independency theorem for superwords. To prove b) it is enough to take the factor by the radical and to use the representability.

□

The item a) of the above corollary was formulated by I.V.L'vov. He also noted the the posibility of reduction of the item b) to a).

G.P.Chekanu [57] made the following observations: 1) the nilpotency condition can be substituted by the quasiregularity of values of some polynomials; 2) not only beginnings of W are linearly independent, but all its subwords also. The periodicity techinique for infinite words allows to obtain a stronger result.

Let W be a maximal nonzero word and let its beginnings are linearly dependent, i.e., the following relation holds

$$\sum_{k=1}^{n} \alpha_k (W)_k = 0.$$

Let $\tilde{W} = \max_{\alpha_k \neq 0} (W)^k$ (let us remind that $(W)^k$ is the end of W, which corresponds to $(W)_k$, i.e., to the beginning of length k, $W = (W)_k (W)^k$). Let $\mathcal{K}$ be the set of k, such that $\alpha_k \neq 0$, $\mathcal{K} \neq \emptyset$. The following proposition holds.

Proposition 2.42 *a) The word $\tilde{W}$ is periodic. There exists a word s of the minimal length, such that $\tilde{W} = s^\infty$.*

b) Let k_0 be the minimal number in $\mathcal{K}$ and $v = (W)_{k_0}$. Then for each $k \in \mathcal{K}$, $(W)_k = v s^{n_k}$, where $n_k = (k - k_0)/|s|$.

c) For m sufficiently big the following relation holds

$$\sum_{k \in \mathcal{K}} \alpha_k v s^{m+n_k} = 0$$

d) The element

$$x = \alpha_{k_0}^{-1} \sum_{k \in \mathcal{K},\ \| > \|,} \alpha_k s^{n_k}$$

is not quasi-invertible.

Proof. The items a) and b) are consequences of Lemma 2.29. If $k \notin \mathcal{K}$, then $(W)_k \tilde{W} = 0$. Hence, by the definition of a zero infinite word, we have that a product of $(W)_k$ and some beginning of $\tilde{W}$ is zero. Hence, a product of $(W)_k$ and some power s^p is zero, because $\tilde{W} = s^\infty$. If we define m as the minimum of such p, then we get the item c). The equality $\sum_{k \in \mathcal{K}} \alpha_k v s^{m+n_k} = 0$ can be written as $vs \sum_{k \in \mathcal{K}} \alpha_k s^{n_k} = 0$. The fact that x is not quasi-invertible, is a direct consequence of this equality. $\square$

Corollary 2.43 (the local independency theorem by G.P.Chekanu) *Let*

(1) a word $W = a_{i_1} \dots a_{i_n}$ is minimal, with respect to the left lexicographic ordering, in the set of all nonzero words of length $\leq n$;

(2) if v_i are ends of W, then for all polynomials $f_i(x) \in xF[x]$, $\deg f_i(x) \leq n$, elements $f_i(v_i)$ are quasiregular.

Then the beginning subwords of W are linearly independent. $\square$

Let us consider the right ideal, which is generated (as a module) by an element u. Let us use the construction 2 from 1.4. By applying the above theorem to the algebra A'', we get the following statement.

Corollary 2.44 (G.P.Chekanu) *If a word W satisfies the conditions (1) and (2) and $uW \neq 0$, then elements $u(W)_i$, $i = 1, \ldots, n$, are linearly independent (by $(W)_i$ is denoted the beginning of W of length i).* $\square$

The following theorem is a consequence of the Corollary 2.43 (see [59]).

Theorem 2.45 (the global independency theorem, G.P.Chekanu) *If a word W satisfies the conditions of the local independency theorem, then all different subwords of W are linearly independent.*

Proof. Let $\sum \lambda_i U_i = 0$ be a linear relation, where U_i are subwords in W. Let us choose in the set $\{U_i\}$ the subset $\mathcal{U}$ of words, which are lexicographically minimal in $\{U_i\}$, and let U_j be the word of the maximal length in the set $\mathcal{U}$. Let $W = SU_jT$, then $0 = \sum \lambda_i SU_i$ and SU_i is either zero, or is a beginning of W (because of the condition (1): the maximality of W in the set of nonzero words). But this contradicts the local independency theorem. $\square$

Remark. In the G.P.Chekanu work was required the quasiregularity of values of polynomials (of degree $\leq n^n$) from the ends of W. Our demand is weaker: we ask for the quasiregularity of values of polynomials (of degree $\leq n$) from ends of W.

In conclusion let us note that the original proofs of the independency theorem ([52], [57]) are rather difficult. We obtained a short and direct proof, because previously we clarified the connection between shifts and the periodicity.

2.1.5 The periodicity and infinite words

The considerations, related to the shift invariancy, lead to the proof of the following combinatorial lemma, which, in its turn, is the foundation of the coincidence proof of the nilradical and the Jacobson radical.

Lemma 2.46 *Let $u \neq v$ be two different subwords in a superword W. Then one of the following three statements holds:*

a) $\exists t : ut \subset W$, $vt \not\subset W$;

b) $\exists t : ut \subset W$, $vt \subset W$; ut and vt are lexicographically comparable. (If u and v are comparable, then we can take the empty word Λ for t);

c) $\exists s : \forall n \in \mathbb{N} \quad s^n \subset W$, and either $u = vs$, or $v = us$.

Proof. Let us suppose that a) and b) don't hold. Then for some nonempty word s, either $u = vs$, or $v = us$. Let us prove that $s^n \subset W$. Let us take t, such that $ut \subset W$, $|t| > 2n|s|$. Then ut and vt are lexicographically comparable and $vt \subset W$.

By the lexicographic comparability of ut and vt, we have that t is a beginning of st, hence, t is a beginning of s^∞. As $|t| > 2n|s|$, then $s^n \subset t \subset W$. I.e., c) holds. $\square$

Lemma 2.47 *Let $u \neq v$ be two different subwords in a superword W. Then one of the following three statements holds:*

a) the word W contains subwords of an arbitrary big length, which don't contain u;

b) $\exists r, t : rut \subset W$, $rvt \not\subset W$;

c) $\exists s : \forall n \in \mathbb{N} \quad s^n \subset W$ and, either $u = vs$, or $v = us$.

Proof. If b) and c) don't hold, then, by the previous lemma, $\exists t : ut \subset W$, $vt \subset W$, ut and vt are lexicographically comparable. Let $u_1 = ut$, $v_1 = vt$, then, after a reordering of generators, we can obtain the relation $v_1 \succ u_1$. Therefore, we reduced the lemma to the case, when $v \succ u$.

Let us note that, if b) doesn't hold, then the subword substitution $u \to v$ preserves the property of a word x to be a subword in W. If a word X is infinite, then this substitution preserves the property "each subword in X is a subword in W".

Let us consider the set $\mathcal{W}$ of right infinite words, such that each subword of the word in $\mathcal{W}$ is a subword in W. The substitution $u \to v$ preserves the membership in $\mathcal{W}$, increasing at the same time the lexicographic order of a word. Therefore, if W' is a maximal word in $\mathcal{W}$ (it exists by Proposition 1.8), then it doesn't contain u.

As each subword in W' is a subword in W, and W', being infinite, contains subwords of any length, then the condition a) holds. $\square$

Corollary 2.48 *Let $\mathcal{U} = \{\sqcap_\rangle\}_{\rangle=\infty}^{\setminus}$ be a finite set of pairwise different subwords of a superword W. Then one of the following three statements holds for each i:*

a) W contains subwords of an arbitrary big length, which, in its turn, don't contain u_i;

b) $\exists r_i, t_i$: $r_i u_i t_i \subset W$, and $\forall j \neq i \quad r_i u_j t_i \not\subset W$;

c) for some j $\exists s : \forall n \in \mathbb{N} \quad s^n \subset W$ and, either $u_i = u_j s$, or $u_j = u_i s$.

This corollary can be proved by the induction on n – the number of subwords in $\mathcal{U}$.

Theorem 2.49 *a) Let W be a u.r. nonperiodic word and $u \neq v$ be its subwords. Then $\exists r, t : rut \subset W$ and $rvt \not\subset W$.*

b) If $\mathcal{U} = \{\sqcap_\rangle\}_{\rangle=\infty}^{\setminus}$ be a finite set of pairwise different subwords of the superword W, then $\forall i$, $1 \leq i \leq n$, $\exists r_i, t_i$: $r_i u_i t_i \subset W$, and $\forall j \neq i \quad r_i u_j t_i \not\subset W$.

c) If $I \neq 0$ is an ideal of the algebra A_W, then I contains a monomial, hence, it contains all sufficiently long monoimials; therefore, the quotient algebra A_W/I is nilpotent. (The definitions of a u.r. word and of an algebra A_W see in 1.6.)

Proof. a) and b) are consequences of those fact that, if an u.r. word W is nonperiodic, then the cases a) and b) of the previous lemmas can be excluded.

Let $\sum \lambda_i u_i \in I$. Then, by the previous corollary, we can choose r_i and t_i, such that $r_i u_i t_i \subset W$ and $r_i u_j t_i \not\subset W$, for all $j \neq i$. Hence, $r_i u_j t_i = 0$ in the algebra A_W and $I \ni r_i(\sum_j \lambda_j u_j)t_i = \lambda_i r_i u_i t_i$. Therefore, I contains a monomial. □

Corollary 2.50 *If a word W is u.r. and nonperiodic, then the Jacobson radical $J(A_W) = 0$.* □

Remark. In Chapter 3 it will be proved that the nilradical in a monomial algebra A coincides with the intersection of all ideals I_W, where W is a u.r. word, such that only finite number of letters in alphabet (i.e., generators) occur in W. There also will be proved that in the periodic case $J(A_W) = 0$. The coincidence of the nilradical and the Jacobson radical in an arbitrary monomial algebra is a consequence of this fact.

2.1.6 The periodicity and the number of subwords

Above was demonstrated the connection between the periodicity and the shift invariancy. Here we shall consider another method of the periodicity proving.

Theorem 2.51 (on the periodicity) *a) If each subword of length n in a superword W uniquely defines the next symbol, then W is periodic with the period $T_W(n)$.*

b) If each subword of length n in a right superword W uniquely defines the next symbol, then W is pseudoperiodic of order $T_W(n)$. The analogous statement holds for left superwords.

(Let us remind that $T_W(n)$ is the number of subwords of length n in a word W and $V_W(n)$ is the number of subwords of length $\leq n$).

Proof. To each subword u of length n in W uniquely corresponds the subword u' of length n, which begins by 1 position to the right from the start position of u. Let us consider an oriented graph, which verteces are subwords of length n and edges are the above correspondences. This graph is connected, has $T_W(n)$ verteces and to the word W corresponds the path, which go through all verteces in the graph. In the case a) this path is a cycle of length $T_W(n)$, in the case b) this path is a cycle with a tail (this tail corresponds to the pre-period). $T_W(n)$ equals to the sum of the cycle and the tail lengths. □

Corollary 2.52 *There exist words u_i of an arbitrary big length and two different letters a and b, such that $au_i \subset W$ and $bu_i \subset W$.* □

By applying the compactness lemma, we have

Corollary 2.53 *The closed orbit of a nonperiodic u.r. word contains two different superwords, such that their right (left) ends coincide.* □

Corollary 2.54 *The conclusion of the theorem holds, if holds any of the following conditions:*

a) $T_W(n+1) = T_W(n)$*;*

b) $T_W(n+1) < n+1$*;*

c) $V_W(n) < n(n+3)/2$*.*

So, if W *is nonperiodic, then* $T_W(n) \geq n+1$ *and* $V_W(n) \geq n(n+3)/2$*, for all* n*.*

Proof. To each subword of length $n+1$ corresponds its beginning subword of length n. If the inequality $T_W(n) = T_W(n+1)$ holds, then this correspondence is a one to one relation. Hence, to each subword u of length n uniquely corresponds the subword w of length $n+1$, such that u is a beginning of w. Therefore, to u uniquely corresponds the subword u' of length n – the end of w.

If $T_W(1) = 1$, then W is periodic, hence, b) is a consequence of a); c) is a consequence of the equalities $n(n+3)/2 = \sum_{i=1}^{n}(i+1)$, $V(n) = \sum_{i=1}^{n} T(i)$. □

The following periodicity theorem is analogous to the previous one.

Theorem 2.55 *a) Let a right superword* W *has two occurences of a word* u *and each subword, which contains* u*, uniquely defines its next symbol. Then* W *is pseudoperiodic. The analogous statement holds for left superwords.*

b) If a superword W *satisfies the conditions of the item a), then its right end is periodic. If a left end of* W *contains infinitely many occurences of* u*, then* W *is periodic and is of the form* $(uc)^\infty$*, where* c *is the sequence of symbols between the successive occurences of* u*.* □

With the help of this theorem we shall prove the continuality of the set of words, which contain the same subwords, as the given nonperiodic u.r.word.

Corollary 2.56 *Let* u *has infinitely many occurences in* W*. Then, either a), or b) holds:*

a) A right end of W *is periodic.*

b) $\exists s_1 \neq s_2; us_1 \subset W, us_2 \subset W$*.* □

Corollary 2.57 *If a superword* W *is prime (i.e., contains each its subword infinitely many times), then, either* W *is periodic, or there exists a function* $k(n)$*, such that* $\forall u \subset W, \exists s_1 \neq s_2 : |s_1| = |s_2| = k(|u|), s_1u \subset W, s_2u \subset W$*.* □

Corollary 2.58 *Let* W *be an u.r. superword and* $u \subset W$*. Then there exist two different words* c_1 *and* c_2 *of the same length, such that* $uc_1 \subset W$ *and* $uc_2 \subset W$*.* □

Theorem 2.59 *Let the item b) of the above corollary holds, then the set of right superwords, each of them consists of subwords of W, is continual.*

We shall need the following proposition.

Proposition 2.60 *Let there be given an infinite tree and each its vertex (except the root) has degree ≥ 3. Then the set of all paths, each of them begins at the root, is continual.* □

A sketch of the theorem proof. Words uc_1 and uc_2 also can be extended to the right by different ways. Verteces of the tree correspond to extensions of u and an arrow join a word and its extension. If we shall extend words, obtained on some step, to the left also, then we shall get a bilateral superword. □

Corollary 2.61 *The closed orbit of each u.r. word is, either finite (the periodic case), or is continual.* □

In the work [80], using the analogous reasoning with branching (see Corollary 2.58), were proved the following results.

Theorem 2.62 *Let W be a nonperiodic u.r. word. Then it contains an infinite (in both directions) chain of lexicographically increasing subwords.* □

Corollary 2.63 *An ω-permutable f.g. nil-semigroup is finite.* □

(A superword, which contains an infinite increasing chain of subwords, is called ω-divided. A semigroup is called ω-permutable, if for each (unilateral) infinite product of its elements $t_1t_2\dots t_n\dots$ there exist an integer k and a non-identical permutation σ, such that $t_1t_2\dots t_k = t_{\sigma(1)}t_{\sigma(2)}\dots t_{\sigma(k)}$.)

2.1.7 A comparison of properties of periodic and almost periodic words

Periodic words are the special case of u.r. words. However, the cases of nonperiodic and periodic u.r. words are greatly different in their properties. Let us begin with the most important thing – with the action of the shift operator T. Periodic words are those words, which are invariant with respect to the action of some power of T, i.e., they are those words, which have a finite T-orbit. An orbit of each nonperiodic word is countable, but its closure can be continual. (One of examples here is the sequence of the first digits of powers of 2). Let us say some words about the rigidity. A sufficienly big segment of a periodic word (with the length not less, two periods) uniquely defines it. However, for each u.r. nonperiodic word there exist two different words in its closed orbit with the same one-halfs (i.e., the subwords, consisting of symbols in nonnegative positions).

On the other hand, a periodic word u^∞ is "softer" in some sense. If in a subword $w \subset u^\infty$ we make the substitution $u \to u^2$, then it will be a subword in u^∞ again. However, if W is a nonperiodic u.r.word and $u \neq v$ are its different subwords, then there exist subwords c_1 and c_2, such that c_1uc_2 is a subword in$W$, and c_1vc_2 is not.

Remark. The periodicity of W can be proved, if we consider shifts of W and their coincidence (except a beginning segment). Then we can use the periodicity theorem. We shall give a sketch of the proof of the following theorem (Shirshov height theorem is a consequence of this theorem and the compactness considerations).

Theorem 2.64 *For an arbitrary superword W one of the following statements holds:*

a) W is n-divided,

b) W can be divided into a finite number of periodic (super)words.

Proof. Let us cut from W a maximal subword u_1 of length n. From the segment to the right of u_1 let us cut a maximal subword $u_2 \neq u_1$. From the segment to the right of u_2 let us cat a maximal subword $u_3 \neq u_2, u_1$ and so on. If this procedure can be performed n times, then the superword W is n-divided. In the opposite case, the right end of W contains less than n different subwords of length n, hence, it is periodic by Corollary 2.54. The periodicity of the left end can be proved analogously.

Remark. Let us note that more constructible variant of the above considerations is the base of Shirshov height proof, which was given by A.T.Kolotov. He obtained the upper bound on the height of order l^{l^n}, where l is the number of generators and n is the degree of the PI-algebra identity. He proved the periodicity with the coincidence of shifts. On the selection round of Moscow Olimpiad both proofs of the periodicity were given.

2.2 Bases in algebras

2.2.1 On the reduction of words to the canonical form

The usual method here is the representation of a word as a linear combination of lexicographically smaller words and the study of nondiminishable words. If a word $t = t_1 \dots t_n$, $t_1 \succ \dots \succ t_n$, is n-divided, for example, then for each nonidentical permutation $\sigma \in S_n$ the word $t_\sigma = t_{\sigma(1)} \dots t_{\sigma(n)}$ is lexicographically smaller, than t. Hence, if in the algebra a polylinear identity of degree p holds, then t is linearly representable by smaller words. Therefore, a nondiminishable word is not n-divided. But we know that the set of all non-n-divided words has a bounded height. In particular, each non-n-divided word of sufficient length contains a power of a subword. This fact is the base of study of Burnside type problems.

It is often enough to find a "good" subword, non necessary in the beginning, but somewhere. It is easier to create a smaller subword somewhere inside. The main idea here, which is the base on the corresponding technique, is that: if somewhere we add something, than in another place he have to make a deletion, so it is enough to look for modifications.

Proposition 2.65 *Let T be a linearly ordered set and $\vec{\tau} = \{\tau_i\}_{i=1}^p$ be a vector in T^p. Then for each permutation $\sigma \in S_p$, either a), or b) holds:*

a) τ_i and $\tau_{\sigma(i)}$ coincide, as elements in T, for all i;

b) $\exists i : \tau_i > \tau_{\sigma(i)}$ and $\exists j : \tau_j < \tau_{\sigma(j)}$. □

Let us note that, if T is only a partially ordered set, then the proposition is wrong, even if we change the coincidence condition in a) for the incomparability condition (in the sense of the T ordering). Therefore, we shall need lemmas about the existense of the linear ordering, with respect to $\succ$, in some sets of subwords. The alternative the pseudoperiodicity $\Leftrightarrow$ the linear order is the base of everything here. Let us formulate corresponding statements, which are consequences of the above proved properties of pseudoperiodic words and the word u^∞ (Proposition 2.3).

Proposition 2.66 *Let e_i be proper beginnings of a noncyclic word u and f_i be its proper ends. Then, if f_1u^2 is comparable with f_2u^2, then $f_1 = f_2$; if u^2e_1 is an end of u^2e_2, then $e_1 = e_2$.* □

Proposition 2.67 *1. Let $|v_i| \leq n$, for all i, and $v_i \neq v_j$, if $i \neq j$. Then, either a), or b) holds:*

a) the set of words $\{v_it\}$ is linearly ordered with respect to the lexicographic ordering;

b) t is quasiperiodic of order $\leq n$.

2. Let us define $(zv)^i$ by the equality $(zv)_i(zv)^i = zv$. Let $|zv| > R > n$, then, either a), or b) holds:

a) $(zv)_R$ is pseudoperiodic of order $\leq n$;

b) $((zv)^i)_R$ constitute a linearly ordered set. □

Proposition 2.68 *If each subword in W of length $|u|$ is a subword in u^∞, i.e., is cyclically conjugate to u, then $W \subset u^\infty$. In other words, the algebra A_u^F is defined by a set of relations of length $\leq |u|$.* □

Let us remind previously proved Lemma 2.6.

The lemma about deletions and addings. *Let $t = t_1vt_2 \subset u^\infty$ and v is cyclically conjugate to u. Then $t_1v^kt_2 \subset u^\infty$, for all $k \geq 0$. In particular, $t_1t_2 \subset u^\infty$ and $t_1v^2t_2 \subset u^\infty$.*

The following proposition was also proved above (see Lemma 2.29).

Proposition. *Let $|z| = R$ and z is pseudoperiodic of order $\leq n$. then z contains a $[R/n]$-th power of a word of length $\leq n$.*

Proposition 2.69 *If z_1 and z_2 are not pseudoperiodic of order $\leq n$, $z_1 \succ z_2$ and z_1 and z_2 are of different n-types, then the n-type of z_1 is greater, than the n-type of z_2.* □

Let us consider permutations now. The following proposition is a direct consequence of Propositions 2.65 and 2.67 about permutations and the linear ordering.

Proposition 2.70 *Let $u = (z)_n$, $e_i f_i = u$, $i = 1, \dots, p$; $z = uy$ and z is not a pseudoperiodic word of order $\leq n$. Let $\sigma \in S_p$, then, either a), or b) holds:*
a) $e_i f_{\sigma(i)} y = uy \quad \forall i$;
b) $\exists i, j : e_i f_{\sigma(i)} y \succ uy \succ e_j f_{\sigma(j)} y$. □

It will be more convenient for us to use the correspondence $\sigma(i) + 1 \to \sigma(i + 1)$, than the correspondence $i \to \sigma(i)$. This correspondence arises under the consideration of the word $t_\sigma = t_0 t_{\sigma(1)} \dots t_{\sigma(p)} t_{p+1}$. The beginning of the word $t_{\sigma(i)}$ corresponds to the end of $t_{\sigma(i+1)}$. Let us reformulate the above proposition.

Proposition 2.71 *In the conditions of the above proposition, let $\{e_i\}_{i=0}^{p}$ be enumerated by numbers from 0 to p, and $\{f_i\}_{i=1}^{p+1}$ by numbers from 1 to $p+1$. We have that $e_i f_{i+1} = u$. Let $\sigma \in S_p$ and let $\sigma(0) = 0$, $\sigma(p+1) = p+1$. Then, either a), or b) holds:*
a) $e_{\sigma(i)} f_{\sigma(i+1)} y = uy \quad \forall i$;
b) $\exists i, j : e_{\sigma(i)} f_{\sigma(i+1)} y \succ uy \succ e_{\sigma(j)} f_{\sigma(j+1)} y$.
c) Also, if $x \succ z \succ y$, then $(x)_{|z|} \succ z \succ y_{|z|}$. □

We shall need the following proposition.

Proposition 2.72 *Let $t = t_0 \dots t_{p+1} \subset u^\infty$ and $|t_i| \geq n = |u|$, for all i, $0 \leq i \leq p+1$. Then, if $t_\sigma = t_0 t_{\sigma(1)} \dots t_{\sigma(p)} t_{p+1} u^\infty$, then t_σ contains subwords α and β, such that $|\alpha| = |\beta| = |u| = n$ and $\alpha \succ u \succ \beta$.*

Proof. By the deletions and addings lemma (see 2.6), we can assume that all t_i are sufficiently long ($|t_i| > 10n$, for example). Let us represent t_i as $t_i = f_i u^{k_i} e_i$, where $k_i > 2$, $e_i f_{i+1} = u$, $|e_i| < n$. If $e_{\sigma(i)} f_{\sigma(i+1)} = u$, for all i, then $t_\sigma = t_0 t_{\sigma(1)} \dots t_{\sigma(p)} t_{p+1} \subset u^\infty$. So, it remains to use the above propositions. □

Let us formulate another useful proposition.

Proposition 2.73 *Let a word z be not pseudoperiodic of order $\leq n$, $u = (z)_n$ and $|z| = K$. Let $zv \subset t_i$, $i = 1, \ldots, p+1$,*

$$t = t_0 \ldots t_{p+1} \subset (zv)^\infty, \quad t_\sigma = t_0 t_{\sigma(1)} \ldots t_{\sigma(p)} t_{p+1} \not\subset (zv)^\infty, \ \sigma \in S_p.$$

Then the word t_σ contains subwords α and β, such that $|\alpha| = |\beta| = |z|$ and $\alpha \succ z \succ \beta$.

Moreover, if α', β', z' are n-types of α, β and z, respectively, then $\alpha' \succ z' \succ \beta'$. □

In Chapter 7 (see [4]) it will be proved the following proposition.

Proposition 2.74 *Let u be a noncyclic word of length n and let f be a polylinear identity of complexity $< n$ and of degree p. Then there exists a word $t \subset u^\infty$, $t = t_0 \ldots t_{p+1}$, such that $2n \leq |t_i| \leq 3n$, for all i, and t is linearly representable by words $t_\sigma = t_0 t_{\sigma(1)} \ldots t_{\sigma(p)} t_{p+1}$, such that $t_\sigma \not\subset u^\infty$.* □

The following proposition is a consequence of Propositions 2.73 and 2.74.

Proposition 2.75 *a) Let f be a polylinear identity of degree p and complexity $< n$, $T(f)$ be the corresponding T-ideal and z be not a pseudoperiodic word of order n. Then the word $(zv)^{2(p+1)}$ is linearly representable modulo $T(f)$ by words, which contain subwords of the form α, $|\alpha| = |z|$, $\alpha \succ z$, and of the form β, $|\beta| = |z|$, $z \succ \beta$.*

Moreover, n-types of α, β and z satisfy the same inequalities. □

2.2.2 Selected sets of words and operations over them

Our aim is to obtain inside the given word the required subword.

Definition 2.76 A set of words $\mathcal{W}$ will be called selected, if the algebra $A/id(\mathcal{W})$ is nilpotent. By $l(\mathcal{W})$ will be denoted the degree of nilpotency. (The combinatorial sense here is that each word of length $\geq l(\mathcal{W})$ is linearly representable by words, which contain a subword from $\mathcal{W}$.) If each word of length $\geq l_k(\mathcal{W})$ is linearly representable by words, which have k occurences of a word from $\mathcal{W}$, then the set $\mathcal{W}$ is called k-selected. By $|\mathcal{W}|$ will be denoted the number of words in $\mathcal{W}$.

Let us note that the factorization by the radical doesn't give new selected sets. Therefore, the search for such sets is essentially the work in the semisimple part, and sometimes such technique allows not to use the structural theory methods. We shall try to obtain the required subword step by step, by creating auxiliary subwords at the beginning. There can be many such steps. Therefore, at first we shall study operations over selected sets. Let us begin with the simplest.

Proposition 2.77 *a) If a set $\{u_i\}$ is selected and $v_i \subset u_i$, for all i, then the set $\{v_i\}$ is selected and $l(\{v_i\}) \leq l(\{u_i\})$.*

b) If a set $\mathcal{A}$ is selected and $\mathcal{B} \supset \mathcal{A}$, then $\mathcal{B}$ is selected and $l(\mathcal{B}) \leq \updownarrow(\mathcal{A})$.

c) A word in a selected set can be replaces by a family of words, by which this word is linearly representable. This operation doesn't increase the degree l.

$\square$

Let us note at first that, if we can create one of words in a set, then we can do it many times.

Proposition 2.78 *If a set α is selected, then it is k-selected and $l_k(\alpha) \leq k|\alpha| \cdot l(\alpha)$.* $\square$

Lemma 2.79 (on circulation) *Let f be a polylinear identity in an algebra A of degree p, $D = \{d_j\}$ be a set of words of degree $\leq p$ (including the empty word Λ) from A generators and elements c_i and $\mathcal{C} = \{\rfloor_\rangle\}$ be a selected set. Then for each positive integer R, the set $\{(c_i d_j)^R, i = 1, \ldots, |\mathcal{C}|, \; | = \infty, \ldots, |\mathcal{D}|\}$ is also selected ($|D|$ is the number of elements in D).*

Proof. Each word of length $\geq l(\{c_i\})$ is linearly representable by words of the form $d_k c_i e_k$, where d_k and e_k are products of generators. Now we can use those fact that each word, of sufficirntly big degree with respect to $\mathcal{C} = \{\rfloor_\rangle\}$, can be, by Shirshov height theorem, represented in the required way (see Proposition 2.35). $\square$

Definition 2.80 Let z be a word and let us denote by z' the following set of words.

If z is pseudoperiodic of order $\leq n$, then $z' = \{z\}$, else $z' = \mathcal{N}_\infty \cup \mathcal{N}_\in$, where $\mathcal{N}_\infty$ is the set of lexicographically smaller non-pseudoperiodic words, which coincide with their n-types (i.e., each proper beginning of a word in $\mathcal{N}_\infty$ is pseudoperiodic of order $\leq n$), and $\mathcal{N}_\in$ is the set of pseudoperiodic words of order $\leq n$ and length R, which are lexicographically smaller, than z. (R is fixed. R will be equal to $2nk$, where k is the required power value.)

Proposition 2.81 *If we shall apply the operation $'$ to the word z and to each words, obtained in this way, in succession (and each time join the obtained sets), then, at last, we shall get, either the set $\{z\}$, or the set $\mathcal{N}_\in$. Moreover, it is enough to make $\leq T(n, R)$ steps, where $T(n, R)$ is the number of n-types, which are not pseudoperiodic of order n and which have length $\leq R$. The set of all n-types of length $\leq R$ also can be reduced to the set of pseudoperiodic words by $T(n, R)$ steps.* $\square$

We already proved that $T(n, R) < Rnl^{n+1}$.

Let us note that, istead of operator $'$, we can use, with the same success, the operation of the construction of the set of smaller words of the same length.

This operator was introduced to make the descent to pseudoperiodic words even more faster. It allows to diminish the estimation from l^{m^2} order to l^m order (l is the number of generators and m is the degree of an algebra). n-types in the heght theorem proof were used only for the improving of estimations also.

So, we established that we can destroy the period and create a both smaller and greater subword. The circulation lemma allows to create a period. Let us sum the results of Proposition 2.75 and Lemma 2.79.

Proposition 2.82 *Let α be a finite selected set, $z \in \alpha$ and z be not a pseudoperiodic word of order $\leq n$. Let the algebra A satisfies a polylinear identity f of degree p and complexity n. Then*

a) if we substitute z by the set of all lexicographically smaller words of the same length, then the new set α' will be also selected;

b) the word z can be replaced by the set z'. □

The following statement is a consequence of Propositions 2.81 and 2.82.

Corollary 2.83 (Shestakov hypothesis) *Let A be a finitely generated PI-algebra of complexity n. Then the set of all k-th powers of words of length $\leq n$ is selected for all k. If all these words are algebraic elements, then A is finite-dimensional.* □

Remark. The same result was obtained above, as a consequence of the independency theorem (Corollary 2.41). However, here we don't use the structural theory (we don't factorize by the radical) and we can get constructible estimations. We don't write these estimations here, because further will be developed the technique, which allows to obtain exponential estimations.

Let us inroduce the following ordering on the set of sets of words: at first the leading words are compared, then the second in order and so on. If the first k leading words in two sets are equal and the leading words in the remaining parts are incomparable, then these two sets are incomparable. If in one set there are no more words, then the set with the greater number of words is greater.

Because the operation $z \to z'$ diminishes z order (we consider sets without multiplicities), then the following statement holds.

Theorem 2.84 *If A is a f.g. PI-algebra of complexity n, then a minimal selected set of words of length R consists of pseudoperiodic words of order $\leq n$.* □

As each noncyclic word has a regular cyclically conjugate word, which corresponds to the leading term of some Lie (Jordan) monomial, then the following statement holds.

Corollary 2.85 *Let us consider a Lie or Jordan algebra A with its inclusion into an associative algebra of complexity n. If all monomials from generators in A of length $\leq n$ are algebraic, then A is locally finite.* □

2.2.3 The pumping over procedure

Lemma 2.86 (the pumping over lemma) *Let A be a PI-algebra with a polylinear identity f of degree m. Let a word W be of the form*

$$W = c_0 v_1 c_1 \dots v_m c_{m+1}$$

where c_i are letters, which don't occur in words v_j. Then W, modulo $T(f)$, can be represented as a linear combination of words

$$W' = c_{i_0} v'_1 c_{i_1} \dots v'_m c_{i_{m+1}}$$

where c_i don't occur in v'_j and not more, than $m-1$ words v'_i, have the length, greater, than $m-1$.

The idea of this lemma is that with the help of the identity almost all symbols from v_i can be gathered into $m-1$ words v'_i.

Proof. We shall need the following auxiliary proposition.

Proposition 2.87 *Let $k_i \geq m$, $k_i = k'_i + q_i$, $k'_i \geq 0$ and $q_j > q_i$, if $j > i$. Then, for each nonidentical permutation $\sigma \in S_m$, the vector $\vec{k}_\sigma = (k'_1 + q_{\sigma(1)}, \dots, k'_m + q_{\sigma(m)})$ is lexicographically smaller, then $\vec{k} = (k'_1 + q_1, \dots, k'_m + q_m)$.*

Proof. If $\sigma(1) \neq 1$, then $\sigma(1) > 1$ and $k'_1 + q_{\sigma(1)} > k'_1 + q_1$. In this case $\vec{k}_\sigma \succ \vec{k}$. If $\sigma(1) = 1$, then we use the inductive descent from m to $m-1$. □

Let us note that an analogous reasoning can be used in the proof of the height boundedness theorem for Lie algebras with a sparse identity (see Proposition 2.162).

Proposition 2.88 *Let us consider the following game. We are given m heaps of some objects. The first player chooses m of these heaps and each of them divides on the right and the left parts. The second player makes a nonidentical permutation of right parts. Then the first player can achieve the situation, such that each heap, except $m-1$ of them, contains not more, than $m-1$ objects.*

Proof. Let us make some ordering of these heaps and let us define the vector, which i-th coordinate is the number of objects in i-th heap. Let us prove that, if the first player cannot increase the vector, then the required situation is achieved.

Indeed, let we have m heaps and $k_1, \dots, k_m$ is the corresponding vector. Let $k_i \geq m$, for all i. Let $k_i = k'_i + q_i$, $q_i = i$, $k'_i = k_i - i$. As $k_i \geq m$, then $k'_i \geq 0$ and we can use the above proposition. □

With the help of this proposition the pumping over lemma can be proved. We play for the first player, when represent W, as a product $W_0 \cdot \ldots \cdot W_{m+1}$, "cutting" words v_i. Then the identity transforms $W_0 \cdot \ldots \cdot W_{m+1}$ in a sum of words, in which W_i are nonidentically permuted. The second player chooses the worst term.

2.2.4 The Kurosh problem and the height boundedness

If all v_i are powers of one element, then we obtain a collecting process. Let $M \subset A$ and let us denote by $M^{(k)}$ the ideal, generated by k-th powers of elements from M. By the pumping over lemma, we have

Proposition 2.89 *Let A be a finitely generated graded associative PI-algebra, $M \subset A$ be a finite set of homogeneous elements, which generate A, as an algebra. If the quotient algebra $A/M^{(m)}$ is nilpotent of degree r, then A is generated, as a linear space, by elements of the form*

$$v_0 m_0^{k_0} v_1 m_1^{k_1} \ldots m_{s-1}^{k_{s-1}} v_s$$

where $|v_i| < r$ and $k_i \geq m$, for all i, not more, than $m-1$ words v_i have length $\geq m$, $m_i \in M$ and there are not more, than $m-1$ equal among them.

In other words, A has a bounded essential height over M (see Definition 2.108).

Proof. Powers $m_i^{k_i}$ can be denoted by letters, then we can apply the pumping over lemma to words v_i. If v_i has length $\geq r$, then it can be linearly represented by words, which contain the m-th power of an element from M. If there are m equal elements among m_i, then we can apply to them the pumping over lemma, by denoting by letters segments between them and the new letters. □

In the work [4] by this method was proved the following statement ("the inversion of the Kurosh problem deduction from the height theorem in the graded case").

Theorem 2.90 *Let A be a finitely generated graded associative (alternating, Jordan) PI-algebra. Let $M \subset A$ be a finite set of homogeneous elements, which generate A, as an algebra, $M^{(k)}$ be the ideal, generated by k-powers of elements from M. Then, if the quotient algebra $A/M^{(k)}$ is nilpotent, for all k, then A has a bounded height over M.* □

The following statement is a consequence of this theorem and Corollary 2.85.

Corollary 2.91 *Let us consider a f.g. Lie or Jordan algebra A with its inclusion into an associative algebra of complexity n. Then A has a bounded height over the set of monomials (from generators) of length $\leq n$.* □

If we remove the "arbitrary" condition, that A is generated by M, then the essential height will be bounded over M.

In the work [4] the pumping over procedure is described in details. In this work it was proved that an analog of the above theorem holds for the class of rings, asymptotically close to associative, i.e., for "good" varieties (alternating and Jordan rings belong to this class). A variety is called good, if the algebra of left multiplications of each f.g. algebra from this variety 1) is finitely generated, 2) is a PI-algebra, 3) has a bounded L-length. The above theorem holds, if a subalgebra, generated by each element, is associative. An algebra A has a bounded L-length, if, for some k, the algebra $L(A)$ of its left multiplications is linearly representable by the set of elements of the form $L(p_1)\dots L(p_q)$, where $q < k$.

Let us note that, in the conditions of Proposition 2.89, we cannot ask only for the nilpotency of $A/M^{(\mathrm{PIdeg}(A))}$. An example of the the algebra $\mathbb{M}_n \otimes xF[x]$ and the set M, which contains one nilpotent element of degree n, demonstrates that the condition of the nilpotency of $A/M^{(\mathrm{PIdeg}(A)-1)}$ is too weak.

The pumping over is a combinatorial analog of the algebraicity reasoning. With the help of this procedure we can easily prove the satisfiability of the Capelli identity and obtain constructible estimations. We shall need the following statement.

Proposition 2.92 *a) Let x, c_i, $i = 0, \dots, m$, be fixed generators. Then the number of words of the form*

$$c_0 x^{k_0} c_1 x^{k_1} \dots x^{k_{r-1}} c_r, \quad \sum k_i = N$$

is a polynomial from N of degree $r - 1$.

b) Let A be a PI-algebra of degree m. Then the dimension of the vector space, generated by words of the form

$$c_0 x^{k_0} c_1 x^{k_1} \dots x^{k_{r-1}} c_r, \quad \sum k_i = N$$

can be estimated by a polynomial from N of degree $m - 1$ (r is fixed). This is the codimension of the T-ideal of A in this subspace.

Proof. The number in a) is $\binom{N}{r-1}$. b) is a consequence of a) and the pumping over lemma. The number of possibilities to choose $m-1$ positions equals $\binom{r}{m-1}$ and the number of possibilities to arrange powers of x in another positions is not more, than $(m-1)^{r-m}$. $\square$

Definition 2.93 Let I be a multiindex $I = (i_1, \dots, i_r)$, $|I| = \sum_\alpha i_\alpha$. An identity of the form

$$f = \sum_{\sigma \in S_{r+1}} \sum_{|I|=N} \lambda_I c_0 x^{i_0} c_1 x^{i_1} \dots x^{i_{r-1}} c_r = 0$$

is called a weak algebraicity identity of order r. An identity of the form

$$f = \sum_{|I|=N} \lambda_I c_0 x^{i_0} c_1 x^{i_1} \dots x^{i_{r-1}} c_r = 0$$

is called an algebraicity identity of order r. If, with the identity f, the identity f_σ holds, for each $\sigma \in S_r$,

$$f_\sigma = \sum_{|I|=N} \lambda_I c_0 x^{i_{\sigma(0)}} c_1 x^{i_{\sigma(1)}} \dots x^{i_{\sigma(r-1)}} c_r = 0,$$

then f is called a strong algebraicity identity of order r. The depth $D(f)$ of a (strong, weak) algebraicity identity f is the maximal power of the variable x, which occures in summands of f.

Obviously, $D(f) < N$ and $\deg(f) = N + r + 1$. The proof of the following statement is analogous to the proof of the pumping over lemma.

Proposition 2.94 *If an algebraicity identity holds, then the word* $c_0 x^{i_0} c_1 x^{i_1} \dots x^{i_{k-1}} c_k$ *is linearly representable by words of the form* $c_0 x^{j_0} c_1 x^{j_1} \dots x^{j_{k-1}} c_k$*, such that not more, than* $r-1$ *numbers from the ordered set* $\{j_\alpha\}$*, can be greater, than* $N-1$. □

The codimension of the space of strong algebraicity identities in not greater, than the product of $r!$ and the codimension of the space of algebraicity identities. Using the previous proposition to compare codimensions and directing N to infinity, we have the following statement.

Proposition 2.95 *a) In each PI-algebra* A *of degree* m *the strong algebraicity identity holds and we can take* $r = m$.

b) If in a PI-algebra a weak algebraicity identity holds, then the strong algebraicity identity of the same order holds also. □

Remark. It is easy to prove that a strong algebraicity identity holds, if $N > r^3$, $r > 2m$. To prove this it is enough to compare the codimension estimations $r!\binom{r}{m-1}(m-1)^{r-m}N^r$ of the space of the strong algebraicity polynomials and $\binom{N}{r-1}$ – the number of words of the above form.

Proposition 2.96 *Let a strong algebraicity identity* f *of the above type holds in some algebra and let* $g(z_1, \dots, z_r, y_1, \dots, y_s)$ *be an arbitrary polylinear polynomial. Then the following identity holds also*

$$f_g = \sum_{|I|=N} \lambda_I g(a_0 x^{i_{\sigma(0)}} b_0, \dots, a_{r-1} x^{i_{\sigma(r-1)}} b_{r-1}, y_1, \dots, y_s) = 0.$$

Proof. The identity f_g can be constructed by grouping terms in g with their permutations σ of the first r variables. □

Let us use the height theorem, the pumping over procedure and the above Proposition. Then we have the following statement.

Corollary 2.97 *Let us consider substitutions in g: words instead of variables. The ordered set (with respect to the order of variables) of such words will be called the substitution set. Then the result of each such substitution is linearly representable by substitutions, such that in their substitution sets all words, except not more, that Nml^{m+1} of them, have length $\leq N^2m^3l^{m-1}$.* □

We now proved the theorem.

Theorem 2.98 *In each finitely generated PI-algebra the Capelli identity of some order holds.* □

Remark. a) Using the above statement and the an estimation on the number of words, which don't contain the N-th power of a word of length $\leq m$, we can get the following estimation on the Capelli identity order: $m^4l^nl^{m^4l^m}$. Let us note that we proved that an identity with a sufficiently wide Young diagram holds also.

b) The Capelli identity can be used in the pumping over procedure with the same success, as the strong algebraicity identity.

Theorem 2.99 *Each finitely based T-ideal I, which belongs to the radical of a finitely generated PI-algebra, is nilpotent.*

Proof. Let g be a sufficienly long product of polynomials from I. By Corollary 2.97, g can be represented as a linear conbination of products of polynomials from I, such that all variables in them, except some bounded quantity, are substituted by short words. As there are finite number of short words, then in each product one of polynomials with the same substitution set occures many times. It remains to note that the ideal, generated by one element from the radical of a f.g. PI-algebra, is nilpotent. This fact is a consequence of Amizur theorem (the radical of a PI-algebra consists of nilpotent elements) and of Shirshov theorem about the boundedness of heights. □

All this simplifies the proof of Razmyslov-Kemer-Braun theorem about the nilpotency of the radical: we can assume that identities, which garantee the algebra finite-dimensionality over the localization of the center, hold. So we have that a product of a radical power and some central polynomial is zero. Then we can take the factor by this polynomial and use the induction on the transcendence degree of the prime factors center and by the complexity.

2.2.5 Sparse identities and the pumping over procedure

The using of sparse identities (the Capelli identity, in particular) is very similar to the using of the strong algebraicity identity. In the same way we can prove the nilpotency of a finitely based T-ideal and also the Capelli identity satisfiability. The following result of A.D.Chanyshev can be easily proved with the pumping over.

Theorem 2.100 (A.D.Chanyshev) *Let all generators in a $\mathbb{Z}$-graded PI-algebra A of degree r be homogeneous of degree 1, and let the equality $x^n = 0$ holds for each homogeneous x. Then A is nilpotent.*

It is known that some sparse identity holds in a PI-alpgebra A. Let m be its degree. We shall need some auxiliary propositions.

The growth in varieties The definition of the growth in varieties is slightly different from the same in words and algebras. Let A be a a relatively free n-generated algebra from $\mathfrak{M}$, then by $G_{\mathfrak{M}}(n)$ will be denoted the dimension of the vector space in it, which is generated by words of length n, such that each letter $x_1, \ldots, x_n$ occurs in this word only once. If $\mathfrak{M}$ is the variety of associative algebras, then $G_{\mathfrak{M}}(n) = n!$. The following result is well known.

Theorem 2.101 *If $\mathfrak{M}$ is the variety of associative PI-algebras, then $G_{\mathfrak{M}}(n) \simeq c(\mathfrak{M})^n$, where $c(\mathfrak{M})$ is a constant.* □

Remark. There exists examples of Lie algebra varieties with an over-exponential growth, i.e., there exist varieties $\mathfrak{M}$, such that $G_{\mathfrak{M}}(n) \simeq (n/c(\mathfrak{M}))!$, where $c(\mathfrak{M})$ is a constant.

With the help of the dimension estimating we have the following corollary.

Corollary 2.102 *If $\mathfrak{M}$ is a variety of associative PI-algebras, then the following sparse identity holds in $\mathfrak{M}$*

$$\sum_{\sigma} \alpha_\sigma y_0 x_{\sigma(1)} y_1 \ldots x_{\sigma(n)} y_n = 0.$$

□

The following statement, which will be used in what follows, is a consequence of Proposition 1.2.

Proposition 2.103 *There exist k and coefficients α_σ, such that for each, poly-linear by x_i, polynomial $F(x_1, \ldots, x_k, y_1, \ldots, y_r)$, the following equality holds*

$$\sum_{\sigma} \alpha_\sigma F(c_1 v_{\sigma(1)} d_1, \ldots, c_k v_{\sigma(k)} d_k, y_1, \ldots, y_r) = 0. \tag{1}$$

This proposition can be proved by considering terms in F with the fixed arrangement of y_i and by applying to these terms Proposition 1.2 and the linearity considerations. □

With the help of this proposition, the following lemma can be proved in the same way, as the pumping over lemma.

Lemma 2.104 (on the sparse pumping over) *Let A be a PI-algebra, in which the equality 1 holds for each F, polylinear by x_i. Let us substitute words v_i for variables x_i. Then $F(v_1,\dots,v_k,\vec{y})$ is linearly representable by elements of the form $F(v'_1,\dots,v'_k,\vec{y})$, where not more, than $k-1$ words v'_i, have length $> k-1$.* □

Remark. The equality $\sum \alpha_\sigma F(x_{\sigma(1)},\dots,x_{\sigma(m)},\vec{y}) = 0$, for all F, is the definition of a sparse identity in the non-associative case (and even in the case of algebraic systems of an arbitrary "arity").

The finite generated case of Chanyshev theorem is a consequence of the height theorem. Let us formulate it in the convenient for us form.

Proposition 2.105 *There exists the function $R(s,n,r)$, such that each s-generated PI-algebra of degree r with homogeneous generators, in which for any homogeneous element x the identity $x^n = 0$ holds, is nilpotent of index not greater, than $R(s,n,r)$.* □

The linearization technique can be applied also to identities, which hold only for homogeneous elements. The reasoning is the same. Let us formulate some auxiliary statements.

Proposition 2.106 *Let $x^n = 0$, for each homogeneous x. Then, if all v_i have the same homogeneous degree, then the following equality holds*

$$\sum_{\sigma\in S_n} v_{\sigma(1)}\dots v_{\sigma(n)} = 0$$

Proof. It is enough to use the equality

$$\sum_{\sigma\in S_n} v_{\sigma(1)}\dots v_{\sigma(n)} = (v_1+\dots+v_n)^n - \sum_k (v_1+\dots+\widehat{v}_k+\dots+v_n)^n +$$
$$+\sum_{i_1<i_2}(v_1+\dots+\widehat{v}_{i_1}+\dots+\widehat{v}_{i_2}+\dots+v_n)^n+\dots+(-1)^{n-1}\sum_i v_i^n.$$

(as usual, the symbol $\widehat{}$ means that the corresponding term is deleted). □

The following proposition is the linearization of Proposition 2.105.

Proposition 2.107 *Let a_{ij} are of the same homogeneous degree, when i is fixed, $i=1,\dots,s$, $j=1,\dots,k$. Let F be a polylinear polynomial from a_{ij} of degree ks, such that each a_{ij} occurs only once in it. Let F be symmetric for each group of variables a_{ij}, when i is fixed. Then, if $ks \geq R(s,n,r)$, then $F(a_{ij}) = 0$.* □

Proof of Chanyshev theorem. By Nagata-Higman theorem, it is enough to check the satisfiability of the identity $x^R = 0$, or, which is the same, the satisfiability of its linearization $\sum_{\sigma \in S_R} v_{\sigma(1)} \dots v_{\sigma(R)} = 0$. But the last statement is a consequence of Propositions 2.105 and 2.107 (s must be equal to $2m$ and R to $R(s,n,r)$). $\square$

Remark. The above theorem can be proved with the help of the generalized algebraicity notion (this way is more complex). The aim is to "pump over" letters inside v_i in the polynomial $F(v_1, \dots, v_m, \vec{y})$. If all v_i contain a sufficienly big power of some x, then the generalized algebraicity can be applied. Using Nagata-Higman theorem, we can achieve that some x_i occurs in v_i in sufficiently big power. If we symmetrize by x_i, then we get the linearization of the generalized algebraicity. This symmetrization adds new terms, where x_i occurs in "wrong" v_j. But we can achieve that lengths of minimal homogeneous components of x_i have a sufficiently fast growth for i (the minimal component in x_i is much greater, than the maximal component in x_{i-1}) and that the passage to an additional term also gives us the pumping over.

2.2.6 The height of algebras and Gelfand-Kirillov dimension

There is a connection between the height theorem and Kurosh problem: "the height boundedness over $Y \Leftrightarrow$ Kurosh problem over Y". Another connection exists in the representable case. As we noted above, the number of elements of the form $y_1^{k_1} \dots y_H^{k_H}$, $\sum k_i \le n$, has the order n^H. Hence, Gelfand-Kirillov dimension of an algebra is not greater, than its height, therefore, by Shirshov theorem, it is bounded in the PI-algebra case. But for representable algebras (hence, by Kemer theorem, for relatively free algebras also) an opposite estimation also holds. In this case Gelfand-Kirillov dimension equals the essential height.

Definition 2.108 A set Y is called an s-base of an algebra A, if there exist a number SH and a finite set $D(Y)$, such that A is generated, as a linear space, by elements of the form $t_1 \dots t_N$, where $N \le 2\,\mathrm{SH} + 1$, and, for all i, either $t_i \in D(Y)$, or $t_i = y_i^{k_i}$, $y_i \in Y$. Moreover, the number of factors $t_i \notin D(Y)$ is not greater, than SH. The minimal number $\mathrm{SH}(A)$, which satisfies the above conditions, is called the essential height of A over Y. For varieties with associative powers the definition is analogous.

Informally speaking, the height boundedness means the possibility of reducing a word to a piecewise periodic form and the height is the number of segments. The essential height is a minimal number of arbitrarily long periodic segments, which are simultaniously necessary for constructing a base of the algebra. The set $D(Y)$ is the finite set of spans between periodic segments (which are "random"). Let us note that, if Y is an s-base of an algebra A and Y generates

A, as an algebra, then "spans" can be expressed by Y elements and A has a bounded height over Y.

If Y and D are given, then the number of elements of the essential height SH, such that the sum of homogeneous powers of leading components t_i is not greater, than N, has the growth of order N^{SH}. So, we have the following statement.

Proposition 2.109 *a)* $\mathrm{GKdim}(A) \leq \mathrm{SH}(A)$.
b) $\mathrm{SH} \geq \overline{\lim}_{n\to\infty}(V_A(n))/\ln(n)$. □

We shall prove the following theorem.

Theorem 2.110 *Let A be a finitely generated representable algebra. Then* $\mathrm{GKdim}(A) = \mathrm{SH}(A)$ *(Gelfand-Kirillov dimension equals the essential height).*

Corollary 2.111 *The essential height of a representable algebra doesn't depend on the choice of Y and its Gelfand-Kirillov dimension is an integer number.* □

Remarks. The integrality of Gelfand-Kirillov dimension of a representable algebra was proved by V.T.Markov. Let us note that there exist PI-algebras with non-integral Gelfand-Kirillov dimension, hence, in the general case, the equality $\mathrm{GKdim}(A) = \mathrm{SH}(A)$ doesn't hold. It will be interesting to construct an algebra, in which the essential height depends on the s-base choice. A polynomial estimation on Gelfand-Kirillov dimension of a PI-algebra (and, hence, a polynomial estimation on the essential height) is a consequence of A.V.Grishin results.

For proving the above theorem we shall need some auxiliary propositions. The following statement is a consequence of Hamilton-Cayley theorem.

Proposition 2.112 *Let A can be embedded into $End_n(R)$, where R is a commutative ring. The each element $x \in A$ satisfies and equality of the following type*

$$x^n + t_1(x)x^{n-1} + \cdots + t_n(x) = 0, \quad \forall i\ t_i(x) \in R$$

□

The following proposition is a direct consequence of the previous.

Proposition 2.113 *Let A can be embedded into $End_n(R)$, where R is a commutative ring. Let $I = (i_1, \ldots, i_k)$ be an multiindex, $|I| = i_1 + \cdots + i_k$. Let for some elements $c_0, \ldots, c_k \in A$, $x_1, \ldots, x_k \in A$ the following relation holds*

$$\sum_{|I|\leq r} \alpha_I c_0 x_1^{i_1} c_1 \ldots x_k^{i_k} c_k = 0, \tag{2}$$

and also hold all relation of the type

$$\sum_{|I|\le r} \alpha_I c_0 x_1^{m_1} x_1^{i_1} c_1 \dots x_k^{m_k} x_k^{i_k} c_k = 0 \tag{3}$$

for all vectors $\vec{m} = (m_1, \dots, m_k)$, $0 \le m_i \le n-1$ *(the coefficients* α_i *don't depend on* $\vec{m}$*). Then for all integers* $M_1, \dots, M_k$ *the following relation holds*

$$\sum_{|I|\le r} \alpha_I c_0 x_1^{M_1+i_1} c_1 \dots x_k^{M_k+i_k} c_k = 0 \tag{4}$$

□

Proposition 2.114 *Let* U *be a linear span of elements of the form* $c_0 x_1^{M_1} c_1 \dots x_k^{M_k} c_k$. *Then the linear span of those elements that one of* M_i *is smaller, than* r, *coincides with* U.

The proof is analogous to the proof of the pumping over lemma. Let us present the idea of it. Let us put in the lexicographic order vectors $(M_1, \dots, M_k)$ and let us demonstrate that elements, which are linearly non-representable by elements with smaller vectors, have the required form. Let us choose in 2 a nonzero term with the maximal multiindex I. Let $m_j = M_j - i_j$. As $M_j \ge r$ and $m_j \ge 0$, then the power $x_j^{M_j}$ can be represented, as $x_j^{m_j} x_j^{i_j}$. Now, using 3, we can represent $c_0 x^{M_1} c_1 \dots x_k^{M_k} c_k$ as a linear combination of terms with smaller power vectors. □

Corollary 2.115 *Let* $S(q)$ *be the dimension of the linear span of elements of the form* $c_0 x_1^{M_1} c_1 \dots x_k^{M_k} c_k$, *such that* $\sum M_j \le q$. *Then, if the system of relations of the (3) type holds, then there exists a constant* C, *such that* $S(q) < C q^{k-1}$.

□

Let us now proceed with the proof of Theorem 2.111. At first, let us estimate dimensions. For some q_0, all c_j and x_j belong to the linear span of words of length $\le q_0$. then $S(q) \le V_A((k+1+kq)q_0)$. Let us note that the satisfiability of the system (3) means the linear dependence of the system of n^k-dimensional vectors with components $c_0 x_1^{m_1+i_1} c_1 \dots x_k^{m_k+i_k} c_k$. These components are enumerated by systems $(m_1, \dots, m_k)$ of nonnegative integers, $m_i < n$. If $i_j \le q$, then the dimension of the space of such vectors is not greater, than $n^k S(kn+q)$, hence, it is not greater, than $n^k V_A((k+1+k(kn+q))q_0) = C_0 V_A(C_1 + C_2 q)$. Let $\underline{\lim}_{n\to\infty} \ln(V_A(n))/\ln(n) < k$, then there exists $\varepsilon > 0$, such that the inequality $V_A(q) < q^{k-\varepsilon}$ holds for infinitely many q. But, then we have that the dimension of the space of the, described above, vectors is less, than $q^{k-\varepsilon/2}$ for some q, which garantee the satisfiability of (3). The satisfiability of (3), for all systems of x_j and c_j, means, by the previous proposition, that the essential height is smaller, than k. So, we have that $\mathrm{SH} \le \underline{\lim}_{n\to\infty} \ln(V_A(n))/\ln(n)$. It remains to use the inequality $\mathrm{GKdim}(A) \ge \underline{\lim}_{n\to\infty} \ln(V_A(n))/\ln(n)$. Theorem is proved.

□

Let us now obtain estimations for the height of a PI-algebra A over the set of words of degree not greater, than the algebra complexity. Let each word of length l is linearly representable by words, such that each of them contains a word from a set α, as a subword. If length of a word is $k \cdot l$, then we can obtain the occurence of k words from α, as subwords. But the estimatimation $k \cdot l$ is too coarse. We cannot improve it without the "pumping over", because otherwise, k subwords can cut the word on segments with lengths $< l$.

Let us consider a relatively free algebra A with generators a_i and c_j, where $j = 1, \ldots, m+1$. We shall denote by u_α and v_β words, which don't contain c_j. Let us assume that in A holds the polylinear identity f of degree m.

Definition 2.116 A word of the form $c_1 v_1 \ldots c_k v_k$, $\sum |v_i| \geq h$, is called a word of width h. A set of words $\{u_\alpha\}$ is called an m-selected of width h, if each word of width h is linearly representable by words, such that each of them contains a word from $\{u_\alpha\}$ as a subword. (c_i can be permuted only in whole, not in parts).

The following statement is a consequence of the pumping over lemma.

Proposition 2.117 *a) Let α be a set of words of width h and l be the maximal length of a word from α. Then each word of width $\geq h + m + l$ is linearly representable by words, such that each of them contains two occurences of words from α.*

b) Each word of width $\geq h + (k-1)(m+l)$ is linearly representable by words, such that each of them contain k occurences of words from α.

Proof. Let us represent the given word of required length by words, such that each of them contains a word from α. In each of these words let us mark this occurence, denote it by c' and perform the pumping over. Then two c_i, which are on the distance $< m$ from each other, let us unite into one. The loss of width will be not greater, than $m + l$. The item b) can be proved by the obvious induction. □

Corollary 2.118 *Let p be the number of words in the set α, which is m-selected of width h. Then, if $h' > h + (kp-1)(m+l)$, then each word of width h' is linearly representable by words, such that each of them contains k occurences of a word from α.* □

Let us remaind the above proved statement.

Proposition 2.35. *Let A be an l-finitely generated PI-algebra of degree m. Let the letter z has $(q+3)ml^{m+1}$ occurences in a word W. Then W is linearly representable by words, which contain a subword of the form $(zv)^q$.*

Let n be the complexity of an algebra, m be its degree, l be the number of generators and p be the minimal degree of a polylinear identity of complexity n. Let $R = 2mn$ and let all quasiperiodic words in a set α have length R. By Proposition 2.81, we have

Proposition 2.119 *The substitution of a word z from α by a word z' increases the m-width by not more, than $((p+3)ml^{m+1}-1)\cdot(m+R)$.* □

The following statement is a consequence of Propositions 2.35, 2.119, 2.118.

Theorem 2.120 *A set, which consists of pseudoperiodic words of order $\leq n$ and of length R, is selected of width*

$$h = ((p+3)ml^{m+1}-1)\cdot Rnl^{n+1}(m+R)$$

□

And, by this theorem and Proposition 2.81, we have

Theorem 2.121 *Let A be an l-generated PI-algebra of complexity n. Let m be the degree of a minimal identity in A (i.e., $m = \deg(A)$) and p be the minimal degree of an identity of complexity n. Then, over the set of words of degree $\leq n$, A has the height*

$$H(l,m,p) = ((p+3)ml^{m+1}-1)\cdot 2mnl^{n+1}m(2n+1)+ml^n.$$

Proof. Let M be the set of all words with length $\leq n$. Each word is linearly representable by elements of the, described in Definition 2.116, form with the same homogeneous degree. If $\sum |v_i| \geq h$, then we can create new m-th power of an element from M and, using the pumping over, we can increase $\sum k_i - s$. The height can be estimated as $s + (\sum |v_i|)/n$. □

Let as note that $H(l,m,p)$ is asimptotically equivalent to pm^2nl^{m+n+2}.

Corollary 2.122 *If an l-generated PI-algebra A has degree $m = \deg(A)$, then, over the set of words of degree $\leq [m/2]$, A has the height*

$$H(l,m) = ((m+3)ml^{m+1}-1)\cdot m^4 l^{m/2+1} + ml^{m/2}.$$

Proof. By Amizur-Levitski theorem, $\mathrm{PIdeg}(A) \leq \deg(A)/2$. □

2.2.7 The height theorem and Kurosh problem

Let A be a PI-algebra and $M \subseteq A$ be its s-base. Then, if all elements in M are algebraic over K, then A is finite-dimensional (Kurosh problem). The following theorem can be considered as a inversion of this statement.

Theorem 2.123 *Let A be a PI-algebra and $M \subseteq A$ be a subset in A, such that each projection $\pi : A \otimes K[X] \to A'$, where the image $\pi(M)$ is integral over $\pi(K[X])$, is finite-dimensional over $\pi(K[X])$. Then M is a s-base of A.*

Remark. a) As the following example will demonstrate, the direct inversion of Kurosh problem is wrong in the non-graded case. Let $A = \mathbb{Q}[x, 1/x]$. Each projection π, such that $\pi(x)$ is algebraic, has a finite-dimensional image. But, the set $\{x\}$ is not a s-base of A. The boundedness of the essential height is a non-commutative generalization of the integrality condition.

b) Let us note that, in the case of Lie PI-algebras, Kurosh problem has a positive solution, but the height theorem is not valid.

c) The theorem can be generalized for good varieties (see Appendix A).

Proof. We shall need some auxiliary statements.

Lemma 2.124 *If an ideal I and the quotient algebra A/I have bounded essential height over (the projection) of M, then A also has a bounded essential height over M.* □

By $|M|$ will be denoted the number of elements in M, $p = \deg(A)$, by a word we understand a word from A generators and elements from M. If D is a finite set of words, then by $l(D)$ will be denoted the maximal length of word in D. The following statement can be derived from the pumping over lemma in the same way, as Theorem 2.120.

Lemma 2.125 *Let each word of length k can be linearly represented by elements of the form $t_1 t_2 \dots t_s$, where, either $t_i \in D$, or $t_i = m_i^{k_i}$, $m_i \in M$, for all i, and $s < 2|M|p + 1$. Then, if $k > 2(2|M|p+1))|D| \cdot l(D)$, then M is a s-base in A.*

The idea of the proof. By substituting a word of length k by word of the above defined form, and, by deleting with the pumping over all $> p$-th powers of elements from M, we shall diminish the word length. The complete proof uses the width notion and is analogous to the proof of proposition 2.117. □

The following proposition is a consequence of Lemma 2.125.

Proposition 2.126 *Let A be a finitely generated algebra and $\pi(M)$ be a s-base in A/I. Then there exists a finite set of elements $\{I_1, \dots, I_p\} \subseteq I$, such that $M \cup \{I_1, \dots, I_p\}$ is a s-base in A.*

Proof. The module A_k, which is generated by words (from A generators and elements from M) of degree $\leq k$, is finite-dimensional for all k. There exist $H \in \mathbb{N}$ and a finite set $D = D(\pi(M))$ (from the definition of the essential height), such that A_k is linearly representable by elements of essential height H with respect to D, modulo a finite linear combination (which depends on k) of elements from I. It remains to choose k sufficiently big (for the condition of the previous lemma to be satisfied) and use the circulation lemma. □

Remark. a) In the homogeneous case it is enough to use only the circulation lemma.

b) This proposition allows to construct inductively the following interesting Shirshov base $M = \cup_{k=1}^{n} M_k$, where n is A complexity, M_1 is the set of A generators and M_i is the set of central polynomials in the algebra of generic s-generated $i \times i$ matrices. Their number equals (see [17]) to the transcendence degree of the center of the algebra of generic matrices and $\operatorname{PIdeg}(\mathbb{M}_i / \operatorname{id}(M_i)) < i$.

Let us come to the theorem's proof.

Step 1. The passage to the factor by the radical. Using the induction on the nilpotency degree of the radical, we assume that the essential height of each f.g. ideal I from $R(A)$ is bounded. If we assume the theorem validity for semiprime algebras, then the condition of Lemma 2.125 will be satisfied modulo such ideal. It remains to use Lemma 2.125. (The more accurate reasoning uses only Amizur theorem about the local nilpotency of the radical).

Step 2. Using the primary decomposition and Lemma 2.124, we come to the case of a prime A.

Step 3. It is known that a prime algebra A can be embedded into a prime algebra A', which is finite-dimensional over its center. Moreover, if A is finitely generated, then $Z(A')$ is finitely generated also, is integrally closed and is the integral closure of A, $A' = A \otimes_{Z(A)} Z(A')$, and for some $h \in Z(A)$, $h \neq 0$, $A'h \subseteq A$. By applying Lemma 2.124 to the ideal $\operatorname{id}(h)$, we pass to the case of an algebra, which is finite-dimensional over its integrally closed center.

Step 4. The reduction to the case of the matrix algebra over an integrally closed commutative ring. There exists a finite integral extension Z' of the center Z, such that $A' = A \otimes_{Z(A)} Z'$ is isomorphic to the matrix algebra $\operatorname{Mat}_n(Z')$ and Z' contains all eigenvalues of elements from M. Let M be a s-base in A'. Then, if a is an arbitrary element in A, then $a = \sum \lambda_i x_i$ is the sum of terms of the essential height H over M, where $\lambda_i \in Z'$ are linearly independent over the field of fractions of Z and $x_i \in A$. In this case all terms with $\lambda_i \notin Z$ can be ignored and a is equal to some x_i.

The end of the proof. Obviously, the condition "M is a s-base in A" is equivalent to the condition "M' is a s-base in Z", where M' is the set of eigenvalues of elements in M. So we come to the commutative case, which validity is a direct consequence of the theorem's conditions.

Remark. "The Kurosh condition" was formulated in terms of the extension $A \otimes K[X]$. We can take an alphabet with one generator for X, because, if M' is not integral over $K[X]$, then some $x \in X$ is not integral over M' in some factor of $K[X]$ of transcendence degree 1. We can take, for example, the function ring on a curve, which pass through a singularity of the element x.

Remark. The question about the local finiteness of algebraic PI-algebras was formulated by A.G.Kurosh. The positive answer to this question was given by I.Kaplansky. His proof was non-constructible. The first combinatorial proof was obtained by A.I.Shirshov, as a consequence of the height theorem. However, this original proof gives only recursive estimation. Further, the estimation was improved by A.T.Kolotov and A.Ya.Belov. In the case of zero characteristic, M.Nagata and G.Higman proved the nilpotency of (not necessary finitely generated) nil-algebras of index n. They also obtained an exponential estimation on the degree of nilpotency. Yu.P.Razmyslov improved this estimation up to n^2. (In the case of positive characteristic, the global nilpotency cannot have place.) The local finiteness of algebraic Lie PI-algebras in the case of zero characteristic was proved by A.I.Kostrikin. E.I.Zel'manov proved this fact in the general case and also proved the global nilpotency of an Engel Lie algebra in the case of zero characteristic.

2.2.8 Algebras over an arbitrary associative-commutative ring

Notations. We shall consider algebras over an associative-commutative ring $\Phi \ni 1$. Such algebra is called a PI-algebra, if in it holds a polylinear identity, such that one of its coefficients equals 1. It will be necessary for us to change the definition of complexity, by ignoring the algebraic radical. Let $\mathfrak{M}$ be a variety of algebras, then by $\overline{\mathfrak{M}}$ will be denoted the variety of algebras, which is defined by the uniform, with respect to each variable, components of identities, which are valid in $\mathfrak{M}$. The complexity $\mathrm{PIdeg}(\mathfrak{M})$ of a variety $\mathfrak{M}$ is the maximal n, such that for some prime factor F of the ring Φ, $M_n^F \in \overline{\mathfrak{M}}$. The equivalent definition is: $\mathrm{PIdeg}(\mathfrak{M})$ is the maximal n, such that an algebra of $n \times n$ matrices with the infinite center belongs to $\mathfrak{M}$. By $\mathrm{alg}(A)$ will be denoted the set of elements in A, which are integral over Φ. Let $\mathrm{GT}(A)$ be the ideal, generated by homogeneous components of identities, which are valid in A; let $\overline{A} = A/\,\mathrm{GT}(A)$ and π be the natural projection; let $R(\overline{A})$ be the radical of $\overline{A}$, $R'(A) = \pi^{-1}(R(\overline{A}))$; let $m = \deg(A)$ and g be an identity, valid in A; let $\widehat{A}_s^F(g)$ be a relatively free s-generated F-algebra in the variety, defined by the identity g; let $T(St_n)$ be a T-ideal, generated by the standart identity of degree n; by $|M|$ will be denoted the number of elements in a set M.

Radical properties of homogeneous components of identities

Theorem 2.127 *Let A be a relatively free Φ-algebra. Then the set of integral over Φ elements is a locally integral ideal* ALG*, such that the factor $A/\,\mathrm{ALG}$ doesn't contain any integral over Φ elements. All homogeneous components of identities belong to* ALG*. If Φ is a Noetherean ring, then tnere exist M, N, K, such that the equality $(x^M - x^N)^K = 0$ holds for each $x \in \mathrm{ALG}$.*

This theorem is a consequence of the following proposition.

Proposition 2.128 *a) Let F be a field, $|F| > \deg(g)$, $F' \rhd F$. Then $\widehat{A}_s^F(g) \otimes_F F' = \widehat{A}_s^{F'}(g)$.*

b) Let x be a homogeneous component of g. Then the center of each prime factor, with a nonzero image of x, contains not more, than $\deg(g)$ *elements, and the factor itself contains not more, than* $\deg(g)^{\deg(g)}$ *elements.*

Let $K = \deg(g)^{\deg(g)}$.

c) If $x \in alga(A)$, then $x^{2K!} - x^{K!} \in Rad(A)$ and there exists N, such that $(x^{2K!} - x^{K!})^N = 0$.

d) $\mathrm{GT}(A) \subseteq \mathrm{alg}(A)$ *and constitutes an algebraic ideal.*

e) Let A be a relatively free algebra, then $\mathrm{alg}(A) = R(\mathrm{GT}(A))$. *If Φ is Noetherean and A is a f.g. algebra, then* $(\mathrm{alg}(A))^t \subseteq \mathrm{GT}(A)$, *for some $t \in \mathbb{N}$.*

Proof. a) If $|F|$ is greater, than the degree of g, then in both $\widehat{A}_s^F(g)$ and $\widehat{A}_s^{F'}(g)$ homogeneous components of g and all their linearizations hold (see [20]). Hence, $\widehat{A}_s^F(g) \otimes_F F' = \widehat{A}_s^{F'}(g)$.

The implications a)⇒b)⇒c)⇒d)⇒e) are obvious. (The implication a)⇒b) is a consequence of the classification of finite simple associative algebras – matrix algebras, over a finite field. The proof of the implication d)⇒e) uses Razmyslov-Kemer-Braun theorem about the nilpotency of the radical). □

Remark. Analogs this proposition and Theorem 2.127 are valid for varieties of non-associative algebras, such that their radical is nilpotent, all simple algebras with a finite center are finite and prime algebras can be embedded in direct sums of simple algebras.

Proposition 2.129 *If the standart identity* St_{2n} *holds in a Φ-algebra A, then A has the, bounded by H, height over the set Y of words of degree $\leq n$. Moreover, H can be estimated as in Theorem 2.121:*

$$H = ((2n+3)2nl^{2n+1} - 1) \cdot 4n^2 l^{n+1}(2n+R) + 2nl^n,$$

where l is the number of A generators.

Proof. We can assume that A is a relatively free algebra in the variety, generated by the standart identity St_{2n}. It is enough to prove that, for each ρ, all words of length ρ in an absolutely free algebra $\mathbb{Z}[x_1, \ldots, x_s]$ are linearly representable, modulo $T(\mathrm{St}_{2n})$, by words $V_{\rho,H}$ of length ρ and height $\leq H$.

Let L be a $\mathbb{Z}$-module, generated by words of length ρ, $M = L \cap [T(\mathrm{St}_{2n}) + \langle V_{\rho,H} \rangle]$, $N = L/M$. Let us note that the estimations in Theorem 2.121 don't depend on the ground field, hence, by the right exactness of the tensor product, we have that for each prime q, $N \otimes \mathbb{Z}_q = 0$. As the $\mathbb{Z}$-module N is finitely generated, then, by Nakayama lemma, $N = 0$. □

The following theorem is a consequence of the above considerations.

Theorem 2.130 *A finitely generated algebra A over an associative-commutative ring Φ with the unit has a bounded height over the set of words* *of degree not greater, than* $n = \mathrm{PIdeg}(A)$. *If* A *is relatively free and* Y *is a Shirshov base in* A, *which consists of words, then for each word* u *of length* $\leq n$, Y *contains a word, which is cyclically conjugate to some power* u^k. *If a set* Y *has this property, then* Y *is an* s*-base in* A, *and, if besides that,* Y *contains all generators of* A, *then* Y *is a Shirshov base.*

Remark. It will be interesting to get a direct combinatorial proof of this theorem, without using the "radical" reasoning. Let us note that the Noetherity of the ground ring Φ was not used.

2.3 Regular words and Lie algebras

The word combinatorial technique in Lie algebras is based on the correspondence between regular words in the enveloping algebra $U(L)$ and leading terms of images of elements from L. Therefore, the problems about the combinatorial analysis of regular words can be stated. This section is devoted to such problems.

Let us remind two definition, which were formulated in the beginning of this work.

Definition 2.131 A word u is called n-encountered in a word W, if W has n non-ovelapping occurences of u. A system of words $\mathcal{U}$ is called k-encountered in a word W, if some word from $\mathcal{U}$ is k-encountered in W.

Definition 2.132 A word is called *non-improvable*, if it cannot be represented as a linear combination of lexicographically smaller words.

2.3.1 Superwords, regular words and the relation $\rhd$

The lexicographic ordering $\succ$ doesn't garantee the linear ordering. Therefore, either the extension of $\succ$ – the ordering $\rhd$ (which was introduced by V.A.Ufnarovski) is considered, or right superwords are studied. In this section we shall prove that the technique of right superwords allows to prove all main properties of this relation. We shall need

Proposition 2.133 *Let* $b \succ a$ *and* W_1, W_2 *be right superwords, such that* $W_2(a,b) \succ W_1(a,b)$. *Then* $W_2(ab,b) \succ W_1(ab,b)$, $W_2(a,ab) \succ W_1(a,ab)$, $W_2(ba,b) \succ W_1(ba,b)$, $W_2(a,ba) \succ W_1(a,ba)$, $W_1(b,a) \succ W_2(b,a)$.

Definition 2.134 $f \rhd g$, if for each two right superwords W_1, W_2, such that $W_2(a,b) \succ W_1(a,b)$, when $b \succ a$, the inequality $W_2(g,f) \succ W_1(g,f)$ holds.

The image of the set of finite words is dense in the set of superwords. Let us note that each finite word uniquely corresponds to the right superword u^∞. To equivalent words correspond the same superwords. The relation $\rhd$ corresponds to the relation $\succ$ on the set of superwords.

Proposition 2.135 *a) The relation $\rhd$ doesn't depend on the choice of W_1 and W_2 and is correctly defined.*

b) In partiqular, $f \rhd g$, if $f^\infty \succ g^\infty$ (i.e., $f^m \succ g^n$, for some m and n).

c) If $f \succ g$, then $f \rhd g$.

d) The relation $\rhd$ is a relation of a linear ordering on the following set of equivalency classes: $f \sim g$, if for some s, $f = s^l$, $g = s^k$.

Proof. Item a) is obvious, item b) is a direct consequence of a). The transitivity of $\rhd$ is a consequence of b). Let us prove c) and d). If f and f are comparable, then all is proved. Otherwise, either $f = gh$, or $g = fh$, and we can use the above proposition and the induction on the length $|f| + |g|$. $\square$

Corollary 2.136 (the betweenness theorem) *Let $f \rhd g$, then $f \rhd fg \succ gf \rhd g$.* $\square$

With the help of the analogous inductive descent the following lemma can be proved.

Lemma 2.137 *Let $uv = vu$, then $u = s^k$, $v = s^n$, for some s.* $\square$

G.Bergman [68] generalized this result. He proved that two commuting elements of a free algebra are polynomials from one element.

Definition 2.138 A word u is called regular, if one of the following equivalent conditions holds:

a) If $u_1u_2 = u$, then $u \succ u_2u_1$.

b) If $u_1u_2 = u$, then $u \rhd u_2$.

c) If $u_1u_2 = u$, then $u_1 \rhd u$.

A word u is called semiregular in the following case: if $u = u_1u_2$, then, either $u \succ u_2$, or u_2 is a beginning of u.

The correctness of this definition is a consequence of the betweenness theorem.

Let us note, that to each nonperiodic word uniquely corresponds a regular word, which is cyclically conjugate to it.

To each regular word u uniquely corresponds a *Lie* monomial, such that the given word is the leading term of this monomial after removing the parentheses. The set of such monomials is called the Hall base. The correspondence is defined by induction: let f is represented, as $f = gh$, where $g \rhd h$ and f and g are regular words. Let Lie monomials are corresponded to g and h. Let, for example, $x \succ y \succ z$. Then to the regular word $xxzxyzxy$ corresponds the following bracket pattern $[[[x[xz]][x[yz]]][xy]]$. It is known that thus obtained regular non-associative words constitute a base of a free Lie algebra L. Hence,

$T_L(n)$ equals to the number of regular words of length n. It can be computed by the inverse Mobius formula [79]:

$$T_L(n) = \frac{1}{n}\sum_{d|n} \mu(d) m^{n/d}.$$

We shall use the following

Proposition 2.139 *Let W be a sequence of left multiplication operators in a Lie algebra L, v be a regular subword in W, $[v]$ be the left multiplication operator on the element, which is produced by the means of the correct arrangement of brackets in the word v. Let $W = cvd$, $W' = c[v]d$, then the maximal word, which can be obtained after removing the parentheses in $[v]$ is W.* □

Now we shall present a collection of statements about regular words from V.A.Ufnarovski survey [55].

Proposition 2.140 *Each word of the form g^n contains the $(n-1)$-th power of a regular word.* □

Definition 2.141 Let $f \gg g$, if, either $f \succ g$, or f and g are incomparable and the length of f is strictly less, than length of the g.

Proposition 2.142 *If f and g are regular words, then $f \gg g \Leftrightarrow f \rhd g$. Moreover, the word fg is regular in this case.* □

Proposition 2.143 *Let h be the maximal in length regular end of f, $f = gh$. Then*

a) for each proper end k of the word f, either $h \sim k$, or $h \rhd k$;

b) the regularity of f is equivalent to the condition that $g \rhd h$. In this case g is also regular;

c) each regular subword in f is, either a subword in g, or a subword in h, or a beginning of f, which intersects h. □

Proposition 2.144 *If ab and bc are regular words and b is nonempty, then abc is a regular word.* □

Proposition 2.145 *If a and b are regular words, then the regularity of ab^n is equivalent to the condition that $a \rhd b$. Hence, the reqularity of ab is equivalent to the regularuty of ab^k, for all k.* □

Theorem 2.146 *Each word f can we uniquely represented as a product $f = f_1 f_2 \dots f_n$, where f_i are regular words and $f_1 f_2 \dots f_n$.* □

2.3.2 Regular words and the height boundedness

Using the finiteness of the number of regular words of length $\leq n$, we have the following theorem.

Theorem 2.147 *a) The set of words, which don't contain a regular subword of length $> n$, has a bounded height over the set of regular words of length $\leq n$. This height equals to the number of such words.*

b) For each n and N, an infinite word W contains, either a regular subword of length $> n$, or the N-th power of a regular word. □

Proposition 2.148 *If u ia a regular word and $|u| \geq k$, then u contains a regular subword u', such that $k \leq |u'| < 2k$.* □

Proposition 2.149 *Let $l \in \mathbb{N}$. Then the set $\mathcal{W}$ of words, for which the collection $\mathcal{R}$ of regular words of length $\leq k$ is not l-encountered, has a bounded height over over the set of regular words of degree $\leq k$.*

Proof. We use the induction on l. By the previous proposition, if a word W contains a regular subword from $\mathcal{R}$, then it contains a regular subword from $\mathcal{R}$ of length $< 2k$. Let $W = W_1 v W_2$, where v is a regular word from $\mathcal{R}$. In the word W_1 the collection $\mathcal{R}$ is not more, than x-encountered, in the word W_2 the collection $\mathcal{R}$ is not more, than y-encountered, and $x + y \leq l - 1$. □

Remark. Actually we also proved that $\mathcal{W}$ has the essential height, which is equal to $(l-1)\times$(the number of regular words of length $\leq k$).

Proposition 2.150 *The product of two regular words g and h is regular, only when the following two conditions hold: a) $g \rhd h$; b) if $g = ab$ and a and b are regular words, then $h \unrhd b$.* □

This result can be generalized (see Theorem 2.159).

Corollary 2.151 *Each superword contains, either the n-th power of a regular word, or a k-divided subword.* □

Lemma 2.152 (on substitution) *Let $\mathrm{Wd}\langle x_1, \ldots, x_s\rangle$ be the set of words over the alphabet $X = \{x_1, \ldots, x_s\}$. Let $x_s \succ \cdots \succ x_1$. Let the letter x_{ij} corresponds to the word $x_i x_1^j$ ($i = 2, \ldots, s$; $j = 0, dots, \infty$). Let $x_{ij} \succ x_{kl}$, if, either $i > k$, or $i = k$ and $j < l$. Let $X' = \{x_{ij}\}$ be the new alphabet. Then*

a) the substitution $x_{ij} \to x_i x_1^j$, which maps $\mathrm{Wd}\langle X'\rangle$ into $\mathrm{Wd}\langle X\rangle$, preserves the relations $\sim$ and $\rhd$. The image of this map is the set of all words in $\mathrm{Wd}\langle X\rangle$, which don't begin from x_1 (and only them);

b) this substitution maps regular words into regular and to the regular arrangement of brackets in an initial word corresponds the regular arrangement in the image of this word (if we make the additional arrangement $[\ldots[x_i x_1]\ldots x_1]$);

c) if $x_{ij} \neq x_{kl}$, then the images of words $x_{ij}x_{rs}$ and $x_{kl}x_{ab}$ are lexicographically comparable.

This lemma is a direct consequence of Propositions 2.140, 2.142, 2.150. $\square$

The process, described in the above lemma, is called the elimination and it is often used in the reduction reasoning. It is important to remember, how many letters were used in the construction of a new variable. Hence, we mark each letter with its uniformity degree. The uniformity degree (the norm) of a word is the sum of powers of letters, occured in this word (multiplicities taking into account), and it is denoted by $\|\ldots\|$.

Let us define now the inverse map from $\mathrm{Wd}\langle X\rangle$ into $\mathrm{Wd}\langle X'\rangle$. x_1 is the lowest letter in the alphabet. Let $W \in \mathrm{Wd}\langle X\rangle; W = x_1^k\overline{W}$, where $k \geq 0$ and $\overline{W}$ doesn't begin with x_1. Then $\overline{W}$ is the image of some word $W' \in \mathrm{Wd}\langle X'\rangle$.

In other words, W' is obtained be deleting of the lowest variable x_1: we delete x_1 from the beginning and arrange inner powers $x_jx_1^k \to x_{jk}$. Let us note that $x_jx_1^k \rhd x_1$ and the lowest letter in W' corresponds to a greater word, hence, the elimination process allows to construct increasing (with respect to the $\rhd$ relation) chains of regular words.

Proposition 2.153 *Let a word W doesn't contain any occurences of x_1^n. Then*
a) $\|W'\| \geq \|W\| - (n-1)\cdot\|x_1\|$, $|W'| \geq |W|/(n+1) - 1$;
b) the maximal norm $\|x_{ij}\|$ of elements from X' is not greater, than $(n-1)\cdot\|x_1\| + \max\|x_i\|$. $\square$

The following statement is a consequence of 2.142.

Proposition 2.154 *Let u and v be lexicographically incomparable regular words. Then the condition $u \rhd v$ is equivalent to the condition "u is a beginning of v".* $\square$

Therefore, the number of regular words, which are incomparable with u and are greater, than u (with respect to the $\rhd$ condition), is not greater, than $|u|-1$ and, hence, is finite. We have the following

Proposition 2.155 *a) An infinite increasing (with respect to the $\rhd$ condition) chain of regular words contains an infinite chain of lexicographically increasing words.*

b) For each function $F(n)$ there exists a function $G(n)$, such that from each increasing chain of length $G(n)$ of regular words u_i: $u_0 \prec u_1 \prec \cdots \prec u_{G(n)}$, such that $|u_0| = 1$ and $|u_i| < F(\max_{j<i}|u_j|)$, for all i, we can select a chain of length n of lexicographically icreasing words. $\square$

The following statement is a consequence of Propositions 2.155 and 2.153.

Corollary 2.156 *There exists a function $H(n,k)$, such that each word v of length $\geq H(n,k)$ over an alphabet of k letters contains, either the n-th power of a regular word, or n lexicographically comparable regular subwords.* $\square$

Definition 2.157 A word W is called *strongly n-divided*, if it is of the form $W = u_1 v_1 \dots u_n v_n u_{n+1}$, where $v_1 \succ v_2 \succ \dots \succ v_n$ and all words v_i are regular.

The following proposition is analogous to Proposition 2.24.

Proposition 2.158 *If a word V contains n pairwise comparable regular subwords and is n-encountered in a word W, then W is strongly n-divided.* □

Theorem 2.159 (E.I.Zel'manov) *There exists a function $G(n,k)$, such that each word of length $> G(n,k)$ over an alphabet of k letters, either contains the n-th power of a regular word, or is strongly n-divided.*

Proof. In a word of length $G(n,k) = n \cdot H(n,k) \cdot k^{H(n,k)}$ some word of length $H(n,k)$ is n-encountered. It remains to apply the previous proposition and Corollary 2.156. □

Remark. The kernel of our reasoning is the elimination of variables. The original proof of Shirshov height theorem has an analogous structure.

The following proposition is a consequence of the above reasoning.

Proposition 2.160 *There exists a function $T(n,k)$, such that each word of length $> T(n,k)$ contains, either n lexicographically comparable regular words, or a subword of the form cu^n, where $c \rhd u$ and c and u are regular words (hence, the word cu^i is regular, for all i).* □

Proposition 2.161 *a) Let $u \succ t$, then $u^k t \succ u^l t$, if $k > l$.*
b) Let $t \succ u$, then $u^k t \succ u^l t$, if $k < l$. □

Proposition 2.162 ("the pumping over") *a) Let $t_i \succ u$ and $k_i \geq n$, for all i. Let a word W is of the form*

$$W = v_0 c u^{k_1} t_1 v_1 c u^{k_2} t_2 \dots v_{n-1} u^{k_n} t_n v_n = \\ = v_0 c u u^{k_1 - 1} t_1 \dots v_{n-1} c u^n u^{k_n - n} t_n v_n.$$

If $\sigma \in S_n$, then let

$$W_\sigma = v_0 c u^{\sigma(1)} u^{k_1 - 1} t_1 v_1 c u^{\sigma(2)} u^{k_2 - 2} t_2 \dots v_{n-1} c u^{\sigma(n)} u^{k_n - n} t_n v_n.$$

Then $W \succ W_\sigma$, for each non-identical permutation σ.
b) Let $u \succ t_i$ and $k_i \geq n$, for all i, and let W be of the form

$$W = v_0 c u^{k_1} t_1 v_1 c u^{k_2} t_2 \dots v_{n-1} c u^{k_n} t_n v_n = \\ = v_0 c u^n u^{k_1 - n} t_1 \dots v_{n-1} c u^1 u^{k_n - 1} t_n v_n.$$

If $\sigma \in S_n$, then let

$$W_\sigma = v_0 c u^{\sigma(n)} u^{k_1 - n} t_1 v_1 c u^{\sigma(n-1)} u^{k_2 - n + 1} t_2 \dots v_{n-1} c u^{\sigma(1)} u^{k_n - 1} t_n v_n.$$

Then $W \succ W_\sigma$, for all $\sigma \neq \mathrm{id}$.

Proof. a) Let us consider the minimal λ, such that $\sigma(\lambda) \neq \lambda$. Then $\sigma(\lambda) > \lambda$ and we can use the previous proposition. The proof of item b) uses the analogous reasoning: the consideration of the maximal λ, such that $\sigma(\lambda) \neq \lambda$. $\square$

Remark. In the "mixed" case the partition of powers u^{k_i}, such that $k_i = k_i' + k_i''$, is constructed from the left to the right: if $t_i \succ u$, then k_i' is the maximal, not previously used, number in the set $\{1, \ldots, n\}$, and, if $u \succ t_i$, then k_i' is the minimal, not previously used, number in the same set. Then, as before, $W \succ W_\sigma$, for all $\sigma \neq \mathrm{id}$.

Definition 2.163 We say that a sparse identity holds in a Lie algebra, if there exist coefficients α_σ, $\sigma \in S_p$, such that the following equality always holds

$$\sum_{\sigma \in S_p} \alpha_\sigma \cdot [y, x_{\sigma(1)}, z_1, \ldots, z_{k_1}, x_{\sigma(2)}, z_{k_1+1}, \ldots, z_{k_2}, x_{\sigma(3)}, \ldots, z_{k_{p-1}}, x_{\sigma(p)}] = 0.$$

Let us sum the obtained results.

Proposition 2.164 *Let a sparse identity of degree n holds in a Lie algebra, and let W be a non-improvable word. Then*

a) this word W is not a strongly n-divided word;

b) a word of the form vs^n, where $v \rhd s$ and v and s are regular words, is not a $2n$-encountered word.

Proof. The item a) is a consequence of Proposition 2.23. If a word W is of the form $W = v_0 c u^{k_1} t_1 v_1 c u^{k_2} t_2 \ldots v_{2n-1} c u^{k_{2n}} t_{2n} v_{2n}$, then there exists a set of n subwords $c u^{k_i} t_i$, such that all t_i satisfy, either the condition of the item a), or the condition b) of Proposition 2.162. With the help of the sparse identity, W can be represented as a linear combination of words W_σ, which are, by Proposition 2.162, lexicographically smaller, than W. But W is non-improvable. We got a contradiction. $\square$

Theorem 2.165 (S.P.Mizshenko) *Each k-generated Lie algebra, in which a sparse identity of degree n holds, has a bounded height over the set of regular words.*

Proof. Let us prove that the set of non-improvable words has a bounded height. By Propositions 2.160 and 2.164, there exist a function $R(n,k)$, such that each regular word of length greater, than $R(n,k)$, is not a $2n$-encountered word. But, then the theorem is a consequence of Proposition 2.149. $\square$

With the help of the same combinatorial technique we can prove the following result due to G.Higman.

Theorem 2.166 *Let W be a superword over the alphabet $\{x_1, \ldots, x_{p-1}\}$, where p is a prime number. Then the sum of all indeces of variables (multiplicities taken into account), either in a beginning subword, or in some regular subword in W is divisible by p.* $\square$

2.3.3 On sandwiches

The superwords technique allows to prove the famous lemma about sandwiches, which is used in the proof of the local finiteness of algebraic Lie algebras.

Definition 2.167 An element x in a Lie algebra is called a sandwich, if $ad(x)^2 = 0$ and $ad(x)\, ad(y)\, ad(x) = 0$, for each y in the algebra.

If the characteristic of the ground field is different from 2, then the second condition is a consequence of the first. It is easy to see that the commutator of two sandwiches is also a sandwich.

Lemma 2.168 (on sandwiches, A.I.Kostrikin) *If a Lie algebra is generated by sandwiches, then it is locally nilpotent.*

Proof. Let $\{y_1, \dots, y_n\}$ be a finite set of sandwiches. Using the compactness lemma and Theorem 2.169 (see below), we can prove that each word U of length n over an alphabet of n letters contains a subword of the form fgf, or of the form f^2, where f and g are regular words. If we make a correct arrangement of brackets inside f and g, then, after the removing of them, the leading word of the result coincides with U. But the correct arrangement of brackets gives us the sandwiches $[f]$ and $[g]$, so we get a zero word in result. Hence, each word of length n in the algebra of left multiplications is linearly representable by lexicographically smaller words. Then we shall get a contradiction, if we take a minimal nonzero word. $\square$

Theorem 2.169 *Each infinite word W contains, either the square of a regular word, or a word of the form fgf, where f and g are regular words.*

Proof. (I.Bakelin.) It is enough to prove the existence of subwords of the form ff or fgf, where f is a semiregular word and g is regular. Indeed, if f' is an end of f, $f' \rhd f$, then f' is a beginning of f, the word ff contains the word $f'f'$, the word fgf contains $f'gf'$, f' is semiregular and $|f'| < |f|$. It remains to use the induction.

Then we can assume that W is a u.r. right superword, which is lexicographically the greatest in the set of all right superwords with the same set of finite subwords. In this case each beginning $(W)_n$ is a semiregular word. The periodic case is obvious, so let W be non-periodic. As W is u.r., then $(W)_n$ is infinitely-encountered in W. For each natural $n \in \mathbb{N}$ let us mark the starting position of the second occurence of $(W)_n$. (The first occurence is in the beginning.) Let b be a letter, which occurs infinitely many times in marked positions. Then bW has the same set of finite subwords, as W. Let us consider one such position. Let $W = (W)_l b (W)_n$, $bW = b(W)_l b(W)_n$. The letter b is a regular word, therefore, if $(W)_l$ is regular, then bW (and, hence, W also) contains a subword of the required form $b(W)_l b$. Otherwise, $(W)_l = (W)_r U$, $U \rhd (W)_l$, and, by the semiregularity of $(W)_l$, U is a beginning of $(W)_l$, i.e., $U = (W)_k$.

By the property of marked positions, $k < n$. Hence, $(W)_k$ is a beginning of $(W)_n$ and W contains the subword $(W)_k b (W)_k$ of the required form. $\square$

Problems of Burside type in Lie PI-algebras can be solved with the help of the sandwiches method. The reasoning here is as follows. By factoring by the maximal locally nilpotent ideal, we come to an algebra without nonzero locally nilpotent ideals. In particular, such algebra has a zero center and such algebra has an exact embedding into the algebra of left multiplications. Then we have to find a Lie polynomial, such that all its values are sandwiches. As its image is an ideal, then, by the sandwiches lemma, this ideal is locally nilpotent, and we get the required contradiction.

2.3.4 On the combinatorial analysis specific character in Lie algebras

In the associative case we have a polynomial growth of f.g. PI-algebras. This fact is a consequence of the height theorem. But, a Lie algebra, generated by generic vector fields on a manifold, is relatively free and can have an intermediate growth: $V_L(n) \simeq c^{\sqrt{n}}$. The simplest such algebra is the algebra of vector fields on the real line. We can assume, that one of those fields is d/dx, then this algebra L can be considered, as the algebra of functions with the bracket $[f,g] = f'g - fg'$. d/dx corresponds to the function $f = 1$. As L doesn't have the exponential growth, then it has relations; as it is generated by generic fields, then it is relatively free and, therefore, is a PI-algebra. On the other hand, the height theorem doesn't hold in L, because it has an intermediate growth (hence, the sparse identity doesn't hold also). Therefore, there exists a non-periodic u.r.superword, which doesn't contain a descending chain of regular subwords, which occured in succession. An example of such subword was given by A.D.Chanyshev.

At first, the specific character of Lie algebras must be studied for Lie algebras of vector fields. The exact description of bases is very difficult even in the associative case, therefore we have to describe bases approximately, i.e., to construct more and more "narrow" sets of elements, which generate the algebra, as a vector space. The other method is to construct bases in "close" extensions. In the algebra of functions with the operation $[f,g] = f'g - fg'$, for example, which is generated by f and 1, components of functions, which are produced from monomials of degree n, are of the form $(f^{(n_1)})^{k_1} \ldots (f^{(n_s)})^{k_s}$, $n_1 > n_2 > \ldots > n_s$, $\sum n_i k_i \leq n$. The intermediacy of the growth of this algebra is a consequence of this formula.

The study of the algebra radical is difficult, because we don't have a classification of simple Lie PI-algebras. We think, that this is impossible to obtain such classification. The fact is that a set of vector fields, which components satisfy a complete system of algebraic differential equations, generates a simple PI-algebra. The problem about the isomorphism of such algebras is, probably, algorithmically unsolvable (we think that this fact can be derived from

Y.M.Matiyasevich result). To obtain a proof of this statement is an important task. Therefore, the classification of simple Lie PI-algebras is possible up to varieties, generated by them (or, which is the same, up to universal attractors in suitable categories of simple Lie algebras). And, at first, Lie algebras of vector fields and their "almost bases" must be studied. It is true that simple algebras correspond to G-structures, which allow a description? Let us rewrite the equality $[f,g] = f'g - g'f$ as $[f,g] = f \odot g - g \odot f$, $f \odot g = f'g = \nabla_g(f)$, where ∇ is a connectedness on the real line. To each plane connectedness ∇ corresponds the operation $\odot$, such that $[f,g] = f \odot g - g \odot f$. This operation can be useful in study of Lie algebras of vector fields. A G-structure, of course, can have a curvature and doesn't admit a plane connectedness. Therefore, the problem arises about embeddings of a Lie PI-algebra into the PI-algebra A^- with the operation $f \odot g = \nabla_f(g)$.

Problems about the growth of algebras are also interesting. Is it true that the growth of a f.g. PI-algebra is not greater, than $c^{\sqrt{n}}$? (For Lie algebras of vector fields on finite-dimensional manifolds it is true.) Also interesting is the problem about the number of subwords in a word, which doesn't contain a descending chain of length n of regular subwords, which occurs in succession.

Probably, to obtain a constructible proof of E.I.Zel'manov results we have to learn, at first to spoil, and then to improve: the naive approach of the direct simplification is without perspectives here.

The notion of the canonical form and many results about the combinatorial analysis in Lie algebras can be generalized to the "supercase" also (see [39], [85], [40], for example).

2.4 The growth of algebras and radical properties

In this section we shall prove the absence of algebras with the growth fuction between linear one and the function $n(n+3)/2$. (Let us remind that this problem can be reduced to the monomial case.) The proof is based on the consideration of relations between radicals and superwords. Let $\mathcal{A}$ be a finite alphabet of more, than one letter. An algebra A is called an algebra of the slow growth, if $V_A(n) = O(n)$.

Definitions. Let W_R be the set of words, which occur in words of A on arbirary big distance from an end, and W_L be the set of words, which occur in words of A on arbitrary big distance from a beginning, and W_{RL} – from both ends simultaneously. Let $T_R(n)$ be the number of words of length n in W_R and $T_L(n)$ and $T_{RL}(n)$ be the same for W_L and for W_{RL}, respectively. Let $J_L = \mathrm{id}(\mathrm{Wd}(A)/W_L)$, $J_R = \mathrm{id}(\mathrm{Wd}(A)/W_R)$, $J_{RL} = \mathrm{id}(\mathrm{Wd}(A)/W_{RL})$.

Obviously, $W_{RL} \subseteq W_R \cap W_L$, $J_R + J_L \supset J_{RL}$, $T_{RL}(n) \le \min(T_L(n), T_R(n))$, for all n, $T(n) \ge \max(T_L(n), T_R(n))$, for all n.

The following example demonstrates that these inclusions are proper.

Example. $A = F\langle a,b\rangle/id(aba, b^2)$. All words in this algebra are of the forms a^n, b, $a^n b$, ba^n, hence, b can be occured in an arbitrary big distance from both ends, but not from both ends simultaneously.

By the finiteness of the set of words of a bounded length, the following statement is true.

Proposition 2.170 *Let A be a f.g. monomial algebra. Then there exists a function $k(n)$, such that for each word u of length $\leq n$ the following condition holds: if u can be occured on the distance $\geq k(n)$ from the right end, then $u \in W_L$, if u can be occured on the same distance from the left end, then $u \in W_R$, if u can be occured on the same distance from both ends simultaneously, then $u \in W_{RL}$.* □

The sets W_{RL}, W_R, W_L can be characterized in terms of superwords. By the compactness considerations (see Lemma 1.32), we have

Proposition 2.171 *W_L is the set of words, which occur in a left superword, W_R – in a right superword, W_{RL} – in a superword.* □

Corollary 2.172 $J_L(A/J_L) = 0$, $J_R(A/J_R) = 0$, $J_{RL}(A/J_{RL}) = 0$. □

Remark. Let us note that the ideals J_L, J_R and J_{RL} are nilpotent and belong to $B_0(A)$ – the union of all nilpotent ideals. But, in a general case, $B_0(A/B_0) \neq 0$.

By the periodicity theorem (Corollary 2.54) and the above considerations, we have

Proposition 2.173 *a) $T_{RL}(n+1) \geq T_{RL}(n)$, $T_R(n+1) \geq T_R(n)$, $T_L(n+1) \geq T_L(n)$.*

b) If the equality $T_R(n+1) = T_R(n)$ holds, then all right superwords are pseudoperiodic of order $T_R(n)$, if $T_L(n+1) = T_L(n)$, then all left superwords are right pseudoperiodic of order $T_L(n)$, and, if $T_{RL}(n+1) = T_{RL}(n)$, then all superwords are pseudoperiodic of order $T_{RL}(n)$. □

Proposition 2.174 *a) If $T_R(n+1) = T_R(n)$, then each word in A is uniquely defined by its beginning subword of length n and its end subword of length $k(n)$. For $T_L(n)$ the analogous statement holds.*

b) If $T_{RL}(n+1) = T_{RL}(n)$, then each word in A is uniquely defined by its beginning and its end subwords of length $n + k(n)$. In this case each word in A is of the form sw_0t, where w_0 is quasiperiodic of order $T_{RL}(n)$ and s and t have length $\leq k(n)$. Each word in A is a product of a word from W_R and a word from W_L.

c) If one of the inequalities $T_{RL}(n+1) \geq T_{RL}(n)$, $T_R(n+1) \geq T_R(n)$, $T_L(n+1) \geq T_L(n)$ becomes the equality for some $n = n_0$, then all this inequalities become equalities for $n > n_0 + k(n_0)$ and the function $T(n)$ is bounded.

d) If $T_{RL}(n_0+1) = T_{RL}(n_0)$, then A has the slow growth, beginning from $2n_0 + 2k(n_0)$; if $T_R(n_0+1) = T_R(n_0)$ or $T_L(n_0+1) = T_L(n_0)$, then A has the slow growth, beginning from $n_0 + k(n_0)$.

e) The conclusion of the item d) holds, if the equality of the form $T(n_0+1) = T(n_0)$ is substituted by the inequality $T(n_0) < n_0 + 1$.

f) If $T_A(n+1) < n+1$, then A has the slow growth.

g) If $V_A(n+1) < n(n+3)/2$, then A has the slow growth.

Proof. The items a) and b) are consequences of the previous proposition, the items c) and d) are consequences of a) and b). The item e) is a consequence of those fact that, if $T(1) \neq 1$ and $T(n) < n+1$, then $T(n+1) \leq T(n)$. The item f) is a direct consecuence of c) and the item g) is a consequence of the equality $\sum_{i=1}^{n}(i+1) = n(n+3)/2$. $\square$

The following theorem is a consequence of Proposition 2.174 b).

Theorem 2.175 *a) If A has the slow growth, then, for n sufficiently big, $T_A(n) \leq T_R(n) \cdot T_L(n)$.*

b) Let $d = \overline{\lim} T(n)$, $e = \underline{\lim} T(n)$, then $e^2 \geq d$.

c) The following limits $\lim_{n\to\infty}(T_A(n) - n)$ and $\lim_{n\to\infty}(V_A(n) - n(n+3)/2)$ always exist. $\square$

3 Uniformly reccurent words and radicals in monomial algebras

In this chapter, for simplicity sake, we shall consider only monomial algebras without unit.

3.1 The nilradical and the Jacobson radical

There are no simple monomial algebras (except the zero algebra), so the correct analog of the simplicity notion is the notion of almost simplicity. A non-nilpotent algebra A is called *monomially almost simple*, if each its factor, by an ideal, generated by a monomial, is nilpotent; A is called *almost simple*, if each its factor is nilpotent. Obviously, each almost simple algebra is monomially almost simple. By a word in A we shall understand a nonzero word, by an infinite word – an infinite word without zero subwords. By k will be denoted the number of letters in the alphabet, under consideration, or the number of generators in the algebra.

Definition 3.1 Let W be an infinite word. Then A_W is an algebra, such that all its relations are of the form $v = 0$, where v is a word, which is not a subword in W.

The following theorem presents a description of all almost simple and monomially almost simple algebras.

Theorem 3.2 *Let A be a monomially almost simple algebra. Then there exists a u.r. word W, such that $A = A_W$. This algebra A is almost simple, only if W is not periodic.*

This theorem is a consequence of Proposition 3.4 below.

Definition 3.3 a) A word c is called *dense encoutered* in an algebra A, if there exist a constant M and arbitrary long nonzero words in A, such that each segment of length M of such word contains c.

b) A word c is called *dense* in an algebra A, if all, sufficiently long, nonzero words in A contain c.

Let us remind that a word is called uniformly recurrent (u.r.), if each its subword is dense encoutered in it. Let us note that, if A is monomially almost simple, then each word in it is a dense encountered word. Indeed, the factor by some word is nilpotent of index n, and this n can be taken, as a constant M from definition.

Proposition 3.4 *a) A word c is dense encountered in an algebra A, only if c is dense in some superword W in A.*

b) This superword W can be considered, as u.r. word.

Proof. Obviously, if c is dense in W, then c is dense in the algebra. The inverse statement can proved in the following way, let us consider a set of words $\{w_i\}$ of unbounded length, such that c occurs in each segment of w_i of length M. Now we can apply the lemma about superwords construction. The item b) is a consequence of Theorem 1.21. □

The following theorem gives us the description of the nilradical in a monomial algebra.

Theorem 3.5 *$N(A)$ is the intersection of all monomially almost simple ideals in A. In other words $x \in N(A) \Leftrightarrow x$ is projected into 0 for each monomially almost simple factor $\Leftrightarrow$ if $x = \sum \lambda_i c_i, \lambda_i \neq 0$, is a representation of x, as a linear combination of words c_i, then each c_i is not a dense encountered word.*

Proof. The theorem is a consequence of the following two propositions.

Proposition 3.6 *Let a word u is not dense encountered, then $u \in N(A)$.*

Proof. If $u \notin N(A)$, then for some $\{c_i\}$ and $\{d_i\}$, the sum $x = \sum \alpha_i c_i u d_i$ is not nilpotent. Let $M = \max(|c_i| + |d_i|) + 2|u|$. Then each power of x is a linear combination of words, such that an arbitrary segment of length M in each of them contains u. □

Proposition 3.7 *Let a word $u \in N(A)$, then u cannot be dense encountered.*

Proof. Let u occurs in each segment of length M of some superword W. Let us represent W as a product of words $c_i u$, such that each beginning segment of $c_i u$ doesn't contain u. As $|c_i| < M$, then there are only finite number of such words c_i. If $x = \sum c_i u$, then $x \in \mathrm{id}(u)$ and x is not nilpotent, because each its power contains a subword, which is also a subword in W, and all terms, which are produced after the removing of parentheses in $(\sum c_i u)^n$, are different. □

Proposition 3.8 *Let A be an almost simple monomial algebra. Then its Jacobson radical $J(A)$ equals zero.* □

Theorem 3.9 *If an infinite word W is periodic with the period n, then A_W is prime and representable. There exists a monomorphism of A_W into the algebra of $n \times n$ matrices over a polynomial ring and also exists an epimorphism of A_W onto the algebra of $n \times n$ matrices over the ground field.*

Proof. The primarity is a consequence of those fact that for each two nonzero words u, v in A_W, there exists a word $w \neq o$, such that $uvw \neq 0$. The construction of the A_W representation see in Chapter 5. □

The algebra A_u, which corresponds to a periodic u.r. word u, is not almost simple, for example, $A_u/\mathrm{id}\{u - u^2\}$ is not nilpotent. However, the following proposition holds.

Proposition 3.10 *If $I \neq 0$ is a homogeneous ideal in A_u, then A_u/I is nilpotent.*

Proof. Let $I \ni s = \lambda_0 c_0 + \sum_{i=1}^{k} \lambda_i c_i$, $|c_i| = |c_0|$, $c_i \neq c_0$, for $i \neq 0$. Each word R in A_u of length $> 2|u| + |c_0|$, is of the form $R = v_1 c_0 v_2$, where $|v_1| \geq |u|$. Hence, $I \ni v_1 s v_2 = \lambda_0 v_1 c_0 v_2 + \sum_{i=1}^{k} \lambda_i v_1 c_i v_2 = \lambda_0 R$, because the last sum, by Proposition 2.3, equals zero. □

The following proposition was proved above (Theorem 2.49).

Theorem. *Let W be a u.r. nonperiodic word. Then, if $I \neq 0$ is an ideal in A_W, then I contains a monomial, hence, I contains all sufficiently long monomials, so the quotient algebra A_W/I is nilpotent.*

Corollary 3.11 *a) If W is a nonperiodic u.r. word, then each ideal in A_W contains a word and A_W is almost simple.*

b) The Jacobson radical of a monomial algebra coincides with its nilradical.

Proof. The item a) is a direct consequence of the above formulated theorem. The item b) is a consequence of Theorems 3.5, 3.2, 3.9, Proposition 3.8 and a). □

Let us note that in the automata case we have $N(A) = J(A) = B(A)$ (see 5.3). In the case of monomial algebras we have only that $N(A) = J(A)$.

3.2 Uniformly recurrent words and the classification of weakly Noetherian monomial algebras

This section is dedicated to the classification of weakly Noetherian monomial algebras. It is known that all left-Noetherian (right-Noetherian) monomial algebras are automata algebras (Corollary 5.39). However, for weakly Noetherian algebras (which satisfy the chain condition for ascending chains of bilateral ideals) it is not so. For example, if W is a u.r. nonperiodic word over a finite alphabet, then the non-automata algebra A_W is finitely generated and doesn't contain any ideals with a non-nilpotent factor (see Theorem 2.49), hence, it is Noetherian.

The description of weakly Noetherian monomial algebras can be done in terms of u.r. words. A family $\mathcal{U}$, which consists of (super)words, will be called Noetherian, if it is finite and each element in $\mathcal{U}$ is either a finite word, or a superword, which is a join of u.r. segments, after a deletion of a finite part. In other words, each element in $\mathcal{U}$ belongs to one of the following types: 1) a finite word;

2) a right superword of the form uW, where $|u| < \infty$ and W is a u.r. superword;

3) a left superword of the form Wu, where $|u| < \infty$ and W is a u.r. superword;

4) a superword of the form W_1uW_2, where $|u| < \infty$ and W_1 and W_2 are u.r. superwords;

5) a u.r. superword.

Theorem 3.12 *A monomial algebra A is weakly Noetherian $\Leftrightarrow$ there exists a Noetherian subset $\mathcal{U}$, such that $A = A_{\mathcal{U}}$.*

The following statement is a direct consequence of this theorem.

Proposition 3.13 *A weakly Noetherian algebra A is an automata algebra, only if all u.r. subwords of superwords from $\mathcal{U}$ are periodic. In this case to finite words correspond segments of the graph, to u.r. words correspond cycles, to words of the type 2 correspond cycles with ingoing segments, to words of the type 3 correspond cycles with outgoing segments, to words of the type 4 correspond pairs of cycles, joined by a bridge.* □

For the proof of Theorem 3.12 we shall need some auxiliary statements.

Lemma 3.14 *Let a monomial algebra B contains a set of nonzero words of the form cv_ic, where $|v_i| \to \infty$ and, for each i, c is not a subword in v_i. Then B is weakly Noetherian.* □

Corollary 3.15 *a) Let A be a weakly Noetherian monomial algebra. Then for each $c \in \mathrm{Wd}(A)$ there exists a constant $\varphi(c)$, such that, for all $u \in \mathrm{Wd}(A)$ of the form $u = cvc$, each segment in u of length $\varphi(c)$ contains c.*

b) If a right superword W has infinitely many occurences of c, then, beginning from some position, each segment in W of length $\varphi(c)$ contains c. The analogous statement is valid for left superwords.

c) If a superword W has infinitely many occurences of c, both to the left and to the right from the given position, then each segment in W of length $\varphi(c)$ contains c. □

So, the infinite encouteredness is equivalent to the dense encounteredness, moreover, the density is bounded from below for all superwords in an algebra. We shall need the following proposition.

Proposition 3.16 *The number of pairwise nonequivalent u.r. superwords in a weakly Noetherian monomial algebra is finite.*

This proposition is a consequence of the following lemma.

Lemma 3.17 *Let $\{W_i\}_{i=1}^{\infty}$ be an infinite sequence of pairwise nonequivalent u.r. superwords. Then for some k there exist a finite word $v \subset W_k$ and an infinite subsequence of superwords $\{W_{i_j}\}_{j=1}^{\infty}$, such that v doesn't occur in W_{i_j}.*

Proof of Proposition 3.16. Let $\{W_i\}_{i=1}^{\infty}$ be an infinite sequence of pairwise nonequivalent u.r. superwords from A. Applying the above lemma to $\{W_i\}_{i=1}^{\infty}$ and then to $\{W_{i_j}\}_{j=1}^{\infty}$ and so on, we get an infinite sequence of words v_i, which don't contain each other. They generate the ideal, which is not finitely generated. □

Proof of Lemma 3.17. Let us suppose the contrary. Then, for each subword v of word from $\{W_i\}_{i=1}^{\infty}$, we can find a number $N(v)$, such that v is contained in all words W_i, $i > N(v)$. By Lemma 3.14, each segment of the superword W_i, $i > N(v)$, of length $\varphi(v)$ contains v. Let u_i be an arbitrary subword in W_i of length i. By the compactness lemma, there exists a superword U, such that each its subword is also a subword in some u_i. For each $v \subset U$ each segment in U of length $\varphi(v)$ contains v. The same is true for each subword c in an arbitrary W_i. Hence, U is uniformly recurrent and contains all subwords of all W_i. But a u.r. word (see Theorem 1.31) has a minimal with respect to the inclusion set of finite subwords. Therefore, U is equivalent to each W_i, hence all W_i are equivalent to each other. We have a contradiction. □

As was proved above, the nilradical $N(A)$ is the intersection of all monomially almost simple ideals. Each such ideal consists of all non-subwords of a u.r. word and, by the above proved statement, the set of all monomially almost simple ideals is finite.

Let $\{c_i\}_{i=1}^{k}$ be generators of $N(A)$ (by the weak Noetherity of A, the set of generators is finite). As c_i cannot be dense encountered, then, by Lemma 3.14, we have the following proposition.

Lemma 3.18 *a) There exists a natural k, such that in each word $v \in \mathrm{Wd}(A)$ we can find a subword u of length k, such that $v = sut$, where s and t are subwords of a u.r. word in A (i.e., s and t don't belong to $N(A)$).*

b) For all sufficiently big k and for each word $v \in N(A)$ of length k, a word from $\mathrm{Wd}(A)$ cannot have more, than one occurence of u. □

(The item b) is a consequence of the quasiperiodicity of solutions of an equation $uw = su$, see Proposition 2.7.)

By the item b) and the weak Noetherity of A, we have

Lemma 3.19 *Let $|u| = k$, $u \in N(A)$. Then the bilateral ideal, generated by u, is finitely generated as a left ideal. The same is true for right ideals.* □

This lemma means the finiteness of the number of branchings in the graph of left (right) multiplications.

For the proof of Theorem 3.12 it remains to use this lemma.

3.3 The Baire radical in monomial algebras. Prime words

Definition 3.20 An infinite word W is called *prime*, if each its subword has infinitely many occurences in W.

Remark. Each u.r. (and, in particular, periodic) word is prime. The inverse statement is not true: a word, which contains any combination of letters, is prime, but is not u.r., because in this word we can find arbitrary long segments of the form b^n (b is a letter), which doesn't contain any other letters. The idea to study prime words is due to T.Gateva.

The following fact is well known.

Theorem 3.21 *A is semiprime $\Leftrightarrow$ the Baire radical $B(A) = 0$.* □

Theorem 3.22 *a) A countably-generated monomial algebra A is prime only when $A = A_W$ for some prime superword W.*

b) A monomial algebra A is semiprime $\Leftrightarrow$ there exists a family of prime words $\mathcal{W} = \{\mathcal{W}_\rangle\}$, such that $A = A_{\mathcal{W}}$.

c) If A is an arbitrary monomial algebra, the A is prime $\Leftrightarrow$ there exists a family of prime words $\mathcal{W} = \{\mathcal{W}_\rangle\}$, such that $A = A_{\mathcal{W}}$, and for each W_i, W_j there exists W_k, such that each subword from W_i or W_j is contained in W_k.

Corollary 3.23 *The Baire radical $B(A)$ is the intersection of all ideals I_W, where W is a prime word.*

Remark. A is monomially prime and a PI-algebra $\Leftrightarrow$ A is of the form A_W for some periodic word $W = u^\infty$.

Proof. If A satisfies the theorem conditions, then no word from A can generate a nilpotent ideal. Then, with the help of the standart reasoning, which use the study of leading terms, at first with respect to the length, and then with respect to the lexicographic ordering, we can easily prove that no element from A can generate a nilpotent ideal. Hence, A is semiprime.

If there exists a family, described in the item c), then each two words have a simultaneous occurence in the given order in some third word. Therefore, the product of ideals, generated by words, is not zero. The consideration of leading terms of a maximal length demonstrates that the product of any two ideals is not zero. Hence, in the case c) A is prime.

It remains to check the existence of required families of words, when A is prime or semiprime.

Let us begin with the item b). Let v is a word in A. It is enough to construct a prime word in A, which contains v. Hence, it is enough to construct a family $\{v_i\}$ of words in A, such that $v_0 = v$, $v_{i+1} = v_i c_i v_i d_i v_i c_i v_i$. As each v_{i+1} is of the form $r v_i s$, then all v_i can be united in a superword v_∞, and each subword in this superword is also a subword in some v_i. As v_i has infinitely many occurences in v_∞, then v also have the same property. Therefore, the superword v_∞ is prime.

So, let v_i be already constructed. As A is semiprime, then $v'_i = v_i c_i v_i \neq 0$, for some c_i, analogously, $v'_i d_i v'_i \neq 0$, for some d_i. Hence, v_{i+1} is constructed. The item b) is proved.

Let us prove a). In this case the number of words in A is countable. Let us enumerate them. Let $\{u_i\}_{i=0}^{\infty}$ be the set of all words in A. We have to construct a prime word, which contains all u_i.

For this it is enough to construct a family of words $\{v_i\}$, such that

$$v_0 = u_0, \quad v_{i+1} = v_i c_i u_i d_i v_i c_i u_i e_i v_i c_i u_i d_i v_i c_i u_i$$

In this case each v_{i+1} is of the form $r v_i s$ and all $\{v_i\}$ can be united in a prime word, which contains each u_i infinitely many times.

Let v_i is already constructed. As A is prime, then $v'_i = v_i c_i v_i \neq 0$, for some c_i. Analogously, $v''_i = v'_i d_i v'_{/}= 0$, for some d_i, and $v''_i e_i v''_i \neq 0$, for some e_i. So, v_{i+1} is constructed. The item a) is proved.

The item c) is a direct consequence of a) and b). $\square$

4 The category of monomial algebras

When we consider the category of monomial algebras with a fixed set of generators, the following question naturally arises. Let two monomial algebras are isomorphic, as algebras. Is it true that they are isomorphic, as monomial algebras, also, i.e., does there exist an isomorphism, which maps generators into generators?

K.Shirayanagi in [93] proved that the answer is positive for finite-dimensional monomial algebras with adjoined unit. We shall prove that the answer is also positive for finitely generated monomial algebras without unit. The same problem for the case of finitely generated monomial algebras with unit and for the infinitely generated case is open.

Theorem 4.1 *Let A and A' be two finitely generated isomorphic monomial algebras without units. Then there exist an isomorphism between A and A', which maps generators into generators.*

Proof. As dimensions of the quotient spaces A/A^2 and $A'/(A')^2$ are equal to number of generators, then A and A' have the same number of generators. The projections of generators constitute F-base in the quotient spaces.

The following proposition means that the set of monomial generators is defined by the maximal set of monomial relations.

Proposition 4.2 *Let A be a monomial algebra with monomial generators $a_1, \dots, a_s$ and let $b_1, \dots, b_s$ be another set of generators. Let $b_i = \sum_j \alpha_{ij} a_j + b_i^*$, where $b_i^* \in A^2$, $\alpha_{ij} \in F$. Let, moreover, $\alpha_{ii} \neq 0$, for all i. Then, for each word $w(x_1, \dots, x_s)$ from $\mathrm{Wd}\langle x_1, \dots, x_s \rangle$, we have that $w(a_1, \dots, a_s) = 0$, if $w(b_1, \dots, b_s) = 0$.*

Proof of Proposition 4.2. The consideration of terms of smaller degrees reduces the proof to the case, when $b_i^* = 0$.

In this case, if we substitute b_i in $w(b_1, \dots, b_s)$ by $b_i = \sum_j \alpha_{ij} a_j$ and remove the parentheses, then the term, which is produced by the substitution $b_i \to \alpha_{ii} a_i$, is proportional to the word, which is different from all other terms. Hence, it is zero. Proposition is proved. □

Let us note that, by the linear independency of projections of b_i in A/A^2, the elements $\sum_j \alpha_{ij} a_j$ are linearly independent, so the conditions $\alpha_{ii} \neq 0$, for all i, can be obtained by a renumeration of generators b_i. By this proposition we have that there exists a set of generators with a maximal, with respect to the inclusion, set of monomial relations.

Let us complete the proof of the theorem. Let us identify the algebras A and A' and let Θ be a set of s generators with a maximal, with respect to the inclusion, set of monomial relations between them. This relations hold in A and A' after some renumeration of generators. (Conversely, monomial relations in A and A' hold for the system Θ.) As all relations in A and A' are monomial, then A and A' are isomorphic. □

Corollary 4.3 *If A and A' are finite-dimensional monomial algebras with adjoined unit, which are isomorphic, as algebras, then they are isomorphic in the category of monomial algebras with a fixed set of generators.*

Proof. The unit is mapped to the unit under an isomorphism and nilpotents are mapped into nilpotents. The sets of nilpotents constitute monomial algebras without units, which are isomorphic, by the above theorem. □

The following statement is obvious.

Proposition 4.4 *In the category of monomial algebras the direct sums are defined. Let $A = \oplus A_i$, then $N(A) = \oplus N(A_i)$, $J(A) = \oplus J(A_i)$, $B(A) = \oplus B(A_i)$, $\mathrm{B}_{\mathrm{ord}}(A) = \sup B_0(A_i)$. (The notion of the Baire order $\mathrm{B}_{\mathrm{ord}}$ will be defined below.)* $\square$

Definitions. Let us define algebras A_α, radicals $B_\alpha \subseteq A$ and projections $\Pi_\alpha : A \to A_\alpha$, $0 \to B_\alpha \to A \xrightarrow{\pi_\alpha} A_\alpha \to 0$ for all ordinals α with the help of the transfinite induction. If $\alpha < \alpha'$, then there exist a monomorphism $I_{\alpha,\alpha'} : B_\alpha \to B_{\alpha'}$ and an epimorphism $\Pi_{\alpha',\alpha} : A_{\alpha'} \to A_\alpha$. All this morphisms will satisfy natural conditions of the commutativity: if $\alpha < \alpha' < \alpha''$, then $I_{\alpha,\alpha''} = I_{\alpha,\alpha'} \cdot I_{\alpha'\alpha''}$, $\Pi_{\alpha'',\alpha} = \Pi_{\alpha'\alpha} \cdot \Pi_{\alpha'',\alpha'}$.

Let $A_0 = A$ and let $B_0(A)$ be a radical, which consists of all elements of algebra, which generate the nilpotent ideal.

Let $\alpha = \alpha' + 1$ be a non-limit ordinal, then let $A_\alpha = A_{\alpha'}/B_{\alpha'}$, where B_α is the inverse image of $B_0(A_\alpha)$ under the natural projection $\Pi_\alpha : A \to A_\alpha$. The maps $I_{\beta,\alpha}$ and $\Pi_{\beta,\alpha}$ are defined in the natural way.

Let α be a limit ordinal. In this case

$$B_\alpha = \varinjlim B_{\alpha'}, \quad A_\alpha = A/B_\alpha.$$

The maps $I_{\beta,\alpha}$ and $\Pi_{\beta,\alpha}$ are defined in the natural way.

Definition 4.5 The minimal α, such that $A_\alpha = A_{\alpha+1} = \ldots = A_\beta$, for all $\beta \geq \alpha$, is called the *Baire order* of A and is denoted by $\mathrm{B}_{\mathrm{ord}}(A)$.

Theorem 4.6 *a) For each ordinal α there exists a monomial algebra A, such that $\mathrm{B}_{\mathrm{ord}}(A) = \alpha$.*

b) A monomial 2-generated algebra A, such that $\mathrm{B}_{\mathrm{ord}}(A) = \alpha$, exists only when the ordinal α is finite or countable.

Proof. a) Let us use the transfinite induction. If α is a limit ordinal, then Proposition 4.4 ensures the inductive step. Let now $\alpha = \alpha' + 1$.

Let $A = A(\alpha')$ be an algebra with the Baire order α'. We shall define $A(\alpha)$ – an algebra with the Baire order α, as an extension by an element c of the conutable direct sum $A' = \oplus A_i$, where all A_i are isomorphic to A. Let $A'' = A' * F\langle c\rangle/I$, where I is the ideal, generated by elements:

1) c^2, cuc, where u doesn't contain c and $u \notin B(A') = B_{\alpha'}(A')$;

2) $cucvchc$, where u and v belong to the same A_i and v is an arbitrary element in A''.

Let $A(\alpha) = A(\alpha')''$.

By the inductive supposition, $\mathrm{B}_{\mathrm{ord}}(A(\alpha')) = \alpha'$. The item a) of the theorem is a consequence of the following proposition.

Proposition 4.7 *a)* $B_{\alpha'}(A(\alpha)) \supseteq \oplus B_{\alpha'}(A_i)$*, for all* α'*.*

b) Let I *be a monomial ideal in* $A(\alpha')$*. Then there exist a monomial ideal* $I'' = \mathrm{id}(\oplus x_i; x_i \in I)$ *and an epimorphism* $A'' \to A''/I''$*.*

c) If $B_{\alpha'}(A) \neq 0$*, then* c *doesn't generate a nilpotent ideal. If* $\alpha'' < \alpha'$*, then the projection of* c *into the algebra* $A''/\,\mathrm{id}(\oplus B_{\alpha''}(A_i))$ *doesn't generate a nilpotent ideal.*

d) $c \notin B_{\alpha'}(A'')$*,* $\mathrm{B_{ord}}(A(\alpha)) > \alpha'$*,* $\mathrm{B_{ord}}(A(\alpha)) \geq \alpha$*.*

e) The projection of c *into the algebra* $A''/\,\mathrm{id}(\oplus B_{\alpha'}(A_i))$ *generates an ideal of the nilpotency degree* 2*, hence* $c \in B_\alpha(A'')$*.*

f) $A(\alpha)/\,\mathrm{id}(c) \cong \oplus A_i$*.*

g) $\mathrm{B_{ord}}(A(\alpha)) \leq \alpha$*.*

Proof of Proposition 4.7. Items a), b), c), e) and f) are obvious. The Item d) is a consequence of c). The item g) is a consequence of e), f) and the inductive supposition. $\square$

Let us come to the proof of the item b) of the theorem. The necessaty of the countability of the ordinal α is obvious. Let us prove the sufficiency.

Let us note that, if α is countable, then, the previously constructed algebra $A(\alpha)$, is countably generated. Hence, it is enough to define an inclusion of a countably generated algebra into a suitable 2-generated algebra, which preserve all radical properties.

So, let A be a countably generated monomial algebra and $\{a_i\}_{i=1}^{\infty}$ be its generators. Let us correspond the empty word to the empty word, words $ba^ib \in \mathrm{Wd}\langle a,b\rangle$ to generators a_i, and let us consider the algebra $A^\wedge = F\langle a,b\rangle/I$, where I is the ideal, generated by the following elements: 1) b^3; 2) aba; 3) words, which correspond to the zero word in the algebra A, under the substitution $ba^ib \to a_i$.

Let us prove that $A(\alpha)^\wedge$ is the required algebra, i.e., $\mathrm{B_{ord}}(A(\alpha)) = \mathrm{B_{ord}}(A(\alpha)^\wedge)$. This statement is a consequence of the following proposition.

Proposition 4.8 *a)* A *can be embedded into* $A^\wedge$*.*

b) To a monomial epimorphism $h : A \to B$*, which maps generators into generators with the same numbers, naturally corresponds the epimorphism of algebras* $h^\wedge : A^\wedge \to B^\wedge$*.*

c) $B_0(A)$ *can be embedded into* $B_0(A^\wedge)$*.*

d) If there is no nilpotent ideals in A*, then there is no nilpotent ideals in* $A^\wedge$*; if* $B_0(A) = 0$*, then* $B_0(A^\wedge) = 0$*; if* $B(A) = 0$*, then* $B(A^\wedge) = 0$*.*

e) $B(A)$ *can be embedded into* $B(A^\wedge)$*.*

f) $B_\alpha(A)$ *can be embedded into* $B_\alpha(A^\wedge)$*.*

g) Let words u *and* v *contain occurences of elements, which correspond to a word from* A*, and let* $uv \neq 0$*. Then, there exist words* s, t, x, y *of the forms:* Λ *– the empty word,* b*,* a^kb*,* tb*,* ba^m*, and the following conditions hold:* $u = su't$*,* $v = xv'y$ *and the words* u'*,* v' *and* tx *correspond to words from* A *(*tx *corresponds to a generator or to the empty word).*

Proof of Proposition 4.8. Items a), b) and g) are direct consequences of definitions. The item d) is a consequence of those fact that each nonzero word in $A^\wedge$ can be extended to a word, which corresponds to a word in A. Let us prove c). Each nonzero word in $A^\wedge$ can have one of the following forms

$$b,\ b^2,\ a^n,\ a^m buba^n,\ a^n bu,\ uba^m,\ u,\ a^n b^2 a^m,\ a^n b,\ ba^m,$$

where $n, m \geq 0$ and u is a word, which corresponds to a nonzero word in A.

Therefore, except the border effects, we have the complete correspondence. Let a word v generates a nilpotent ideal in A and let u corresponds to v. Then a nonzero word in $A^\wedge$, which has n different occurences of u, is of the form srt, where s has one of the following forms: Λ – the empty word, b, $a^k b$; t has one of the following forms: Λ, b, ba^m; and r corresponds to a word in A, which has n different occurences of v. Hence an element from B_0 corresponds to an element from B_0. Item c) is proved.

Items d) and e) are consequences of the transfinite induction reasoning.

So, the proposition, and the theorem also, are proved. □

5 Automata algebras

The main object of study in this chapter is automata (monomial) algebras. The notion of an automata algebra is the natural generalization of the notion of a finitely generated monomial algebra. Almost all results, which are valid for finitely generated monomial algebras, are also valid for automata algebras.

Everywhere in this chapter the term "algebra" means a monomial algebra. Moreover, we shall assume that each algebra has the unit, which is represented as the empty word (from generators).

Let us remind some well known definitions from the finite automata theory. Let we are given an alphabet (i.e., a finite set) X. By finite automaton (FA) with the alphabet X of input symbols we shall understand an oriented graph, which edges are marked with the letters from X. One of the verteces of this graph is marked, as initial, and some verteces are marked, as final. A word w in the alphabet X is called accepted by a finite automaton, if there exists a path in the graph, which begins at the initial vertex and finishes in some final vertex, such that marks on the path edges in the order of passage constitute the word w.

By a language in the alphabet X we understand some subset in the set of all words (chains) in X. A language L is called regular or automata, if there exists a finite automaton, which accept all words from L and only them.

There are several variants of definition of a finite automaton. An automaton is called determinate, if all edges, which start from one vertex are marked by different letters (and there are no edges, marked by the empty chain). If we reject such restriction and also allow edges, marked by the empty chain, then we shall come to the notion of a non-determinate finite automaton. Also we can allow an automaton to have several initial verteces. The following result from the finite automata theory is well known: for each non-determinate FA there exists a determinate FA, which accepts the same set of words.

It will be convenient for us to consider the class of non-determinate FA, which are produced from determinate, by marking all verteces, as initial and final simultaneously. The reason of this is that the language of nonzero words in a monomial algebra has the following property: each subword of a word, which belongs to the language, is also belongs to it.

Definition 5.1 Let A be a monomial algebra (not necessary finitely defined). A is called an automata algebra, if the set of all its nonzero words from A generators is a regular language.

Obviously, a monomial algebra is an automata algebra, only if the set of its nonzero words is the set of all subwords of words of some regular language.

Let us give another (equivalent) description of automata algebras.

Definition 5.2 Let u be a nonzero word in an algebra A. A word v is called an extension of u, if $uv \neq 0$.

Words u and w in A are equivalent, if the set of all extensions of u coincides with the set of all extensions of w.

Proposition 5.3 *A monomial algebra is automata, only if the set of all its nonzero words has a finite number of equivalency classes.*

Proof. Obviously, the set of equivalency classes is finite, if the algebra is automata. Let the number of classes is finite. We shall give a construction of a minimal determinate automaton of the algebra A.

Verteces: equivalency classes of nonzero words in the algebra A.

Edges: a vertex $\{u\}$ is connected by an edge with a vertex $\{v\}$, which is directed from u to v and is marked by a, where a belongs to the set of A generators, if $ua \sim v$ (i.e., the word ua belongs to the equivalency class of the word v).

The initial vertex: the equivalency class of the empty word.

The set of final verteces: each vertex is final.

It is easy to check that this definition is correct and that the language, which is defined by thus constructed finite automaton, coincides with the set of nonzero words in A. Moreover, this finite automaton also has the minimality property: each two its verteces are nonequivalent, i.e., there exists a path, which begins in one of these verteces, such that there doesn't exist a path, which begins in another vertex, with the same marks on edges. (Different verteces, if we consider them as initial, generates different languages.) $\square$

Proposition 5.4 *A finitely defined monomial algebra is automata.*

Proof. Let the maximal degree of defining relations of the algebra is n. Then the set of extensions of each nonzero word of length $\geq n-1$ is uniquely defined by its end of length $n-1$. Hence, the number of equivalency classes of nonzero words is not greater, than the number of nonzero words of length $\leq n-1$. $\square$

The inverse statement is wrong: the class of automata algebras is broader, than the class of finitely generated monomial algebras. Let us consider, for example, a monomial algebra with three generators $\{a, b, c\}$ and infinite number of relations $\{ab^n c = 0, n = 0, 1, \ldots\}$. This algebra is automata, but not finitely defined.

Definition 5.5 A superword W is called automata, if the set of its subwords constitutes a regular language.

Proposition 5.6 *The following properties of a superword W are equivalent:*

a) W is automata;

b) W can be embedded in the above defined graph;

c) the set of subwords in W has a finite number of equivalency classes, such that the substitution of a subword s in a word v, by an arbitrary subword in the same class, preserves the property of "being a subword" in W. $\square$

Proposition 5.7 *a) A u.r. word is automata $\iff$ it is periodic.*

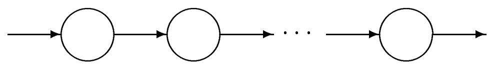
b) A prime automata superword can be embedded into a strongly connected graph.

c) A prime automata algebra is of the form A_W, where W is a prime automata superword, which can be embedded into the graph of this algebra. $\square$

5.1 Growth functions of automata algebras

Let $\Gamma(A)$ be a minimal determinate graph of an automata algebra A. A vertex is called cyclic, if there exists a path, which begins and ends in this vertex. A vertex is called twice cyclic, if there exist two different paths, which begin and end in this vertex and which don't pass through any other vertex twice. In other words, a vertex is twice cyclic, if it belongs to two different cycles. A cycle is a subgraph in the given graph, which contains a path with the following properties: 1) it begins and ends in the same vertex; 2) it doesn't pass through any other vertex twice.

Let a graph Γ doesn't contain any twice cyclic verteces. A chain is a subgraph in Γ, which consists of the sequence of edges, such that: 1) the end of one edge is the beginning of the next; 2) any vertex can occur in this sequence only once. By a simple graph in Γ we shall call a subgraph, which consists of a finite number of cycles, enumerated by numbers $1, 2, \ldots, d$ and such that any pair of adjacent cycles with numbers $i, i+1$ is connected by exactly one chain, which is directed from the i-th cycle to the $i+1$-th. There can be one chain, which is ingoing into the first cycle, and there can be one chain, which is outgoing from the last cycle. The number d of cycles is called the length of a simple graph.



A simple subgraph is a minimal subgraph (in a graph without twice cyclic verteces), which contains a path.

Theorem 5.8 (V.A.Ufnarovski) *Let A be an automata algebra and $\Gamma(A)$ be its minimal determinate graph.*

1) If $\Gamma(A)$ has a vertex, which belongs to two different cycles, then A has an exponential growth function.

2) If $\Gamma(A)$ doesn't have any twice cyclic verteces, then A has a polynomial growth function. The power of the growth (Gelfand-Kirillov dimension) equals to the number of cycles in the maximal simple subgraph in $\Gamma(A)$.

Proof.

1) Let v be a vertex, which belongs to two different cycles C_1 and C_2, let u_1 be the word, which corresponds to the path from v to v along C_1 (i.e., which consists of marks on C_1 edges in the order of passage), and let u_2 be the word, which corresponds to C_2. The item 1) is a consequence of those fact that u_1 and u_2 generate a free rank 2 subalgebra in A.

2) Let us denote by d the length of a maximal simple subgraph. Let $V(n)$ be the number of all nonzero words of length $\leq n$ in A. It is enough to prove that there exist numbers c_1 and c_2, such that the inequality

$$c_1 n^d \leq V(n) \leq c_2 n^d$$

holds for all sufficiently big n (c_1 and c_2 are independent from n).

Let us prove the upper estimation. The graph $\Gamma(A)$ can be represented as a join of a finite number of simple subgraphs (maybe intersecting), such that each path in $\Gamma(A)$ is completely contained in some subgraph. Therefore, it is enough to prove the inequality for a subgraph. Let H be a simple subgraph, which contains k cycles, $1 \leq k \leq d$. We shall construct a simple graph H', with the same number of cycles and chains, which connect cycles, such that all its cycles have the length 1 (i.e., they are loops) and all simple paths also have the length 1 (i.e., they are edges). Obviously, the number of paths of length n in H' is not less, than the number of such paths in H, hence, it is enough to prove the inequality for a simple graph, such that all its cycles and chains have the length 1. But for such simple graph the number of paths of length $\leq n$ is easy to compute. Each path is uniquely defined by the number of rotations in each cycle and also by the occurence, or nonoccurence, of the edge before the first cyclic vertex and the edge after the last cyclic vertex, in this path. The sum of rotations in all cycles is $\leq n$. The number of all representations of n as a sum of d summands, with regard to their order, is $\binom{n+d}{d-1} \leq Kn^{d-1}$, where K is a constant, independent from n. The number of sums, not greater than n, is $\leq n \cdot Kn^{d-1} = Kn^d$. Hence, the number of all paths of length $\leq n$ is not greater, than $3Kn^d$.

The lower estimation can be proved analogously. □

Proposition 5.9 *The Gilbert series of an automata algebra is rational.* □

5.2 Matrix representations and polynomial identities of automata algebras

In this section the following two theorems will be proved.

Theorem 5.10 *An automata monomial algebra can be embedded into a matrix algebra over a free algebra.*

Theorem 5.11 *Let the graph of an automata monomial algebra A doesn't have verteces, which belong to two cycles. Then A can be embedded into a matrix algebra over a field.*

By Theorem 5.11 and results of the previous section, we have

Corollary 5.12 *Let A be an automata monomial algebra and $\Gamma(A)$ be its minimal determinate graph. the following conditions are equivalent:*
(1) $\Gamma(A)$ doesn't have any twice cyclic verteces;
(2) A has a polynomial growth;
(3) A has a non-exponential growth;
(4) A can be represented by matrices over a field;
(5) a polynomial identity holds in A.

Proof of Corollary 5.12. The equivalency (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) was proved in Theorem 5.8. The implication (4) $\Rightarrow$ (5) is obvious. The implication (1) $\Rightarrow$ (4) is a consequence of Theorem 5.11. So we have to prove the implication (5) $\Rightarrow$ (1). Indeed, if a polynomial identity holds in A, then $\Gamma(A)$ cannot have any twice cyclic verteces, because, otherwise, A would contain a free rank 2 subalgebra (see the proof of Theorem 5.8). We can also note that the growth of an automata algebra, which graph contains a twice cyclic vertex, is exponential, and the growth of each finitely generated PI-algebra is polynomial (by Shirshov theorem about the height boundedness). □

To prove Theorems 5.10 and 5.11 we shall present a direct construction of a representation of an automata algebra.

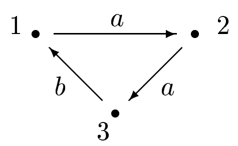
5.2.1 The construction of representations of automata algebras

Let us enumerate verteces of the graph $G(A)$ of an algebra A by numbers $1, 2, \dots, n$, and let $a_1, \dots, a_m$ be generators of A. Let us denote by $\tilde{t}_{k,ij}$ *free variables*, which generate the free algebra $K\langle\tilde{t}_{k,ij}\rangle$, where $k = 1, \dots, m$, $i, j = 1, \dots, n$ (i.e., k runs through the algebra generators and i, j – through the verteces of $G(A)$). Respectively, by $t_{k,ij}$ we shall denote *free commutative variables*, which generate the algebra $K[t_{k,ij}]$ of commutative polynomials. This variables correspond to $G(A)$ edges: a variable $t_{k,ij}$ corresponds to the edge, which connects the verteces i and j and is marked by the letter a_k. Let us consider the following two homomorphisms from A, which are defined by the images of generators. The both two homomorphisms are extensions of semigroup homomorphisms of the semigroup of A nonzero words into a matrix semigroup.

- The homomorphism φ from A into the algebra of $n \times n$ matrices over a free algebra:
$$\varphi(a_k) = \sum_{(i,j)} \tilde{t}_{k,ij} e_{ij} \ ,$$
where the summation is taken over all pairs (i, j), such that in $G(A)$ there exists an edge, which is directed from the vertex i to the vertex j and is marked by a_k. By e_{ij} the matrix units are denoted.

- The homomorphism ψ from A into the algebra of $n \times n$ matrices over an algebra of commutative polynomials:
$$\psi(a_k) = \sum_{(i,j)} t_{k,ij} e_{ij} \ ,$$
where the summation is taken over the same pairs of indeces, as in the previous case.

Example. Let us consider an automata algebra A, such that all its nonzero words are subwords in the infinite cyclic word $aabaab\ldots$. Algebra A can be represented by the following graph $G(A)$:



The reprsentation ψ maps A into the algebra of 3×3 matrices over the polynomial ring $K[t_{a,12}, t_{a,23}, t_{b,31}]$ and is defined by the matrices

$$\psi(a) = \begin{pmatrix} 0 & t_{a,12} & 0 \\ 0 & 0 & t_{a,23} \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(b) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t_{b,31} & 0 & 0 \end{pmatrix}.$$

We defined the maps φ and ψ on generators of the algebra A. Let us naturally extend this maps to words from generators. Obviously, the images of zero words in A are equal to zero (see Lemma 5.13). Hence, the maps φ and ψ can be extended to homomorphisms of the semigroup of nonzero words in A into the matrix semigroup, therefore, they can be extended to homomorphism of the semigroup algebra into the matrix algebra. So φ and ψ define homomorphisms of A into the matrix algebra.

The homomorphism ψ will be called the *canonical representation* of an automata algebra A. Let us note that to each embedding of a word into the graph corresponds the operator of the form λE_{ij}, where i is the number of the initial vertex of the word, and j is the number of the last vertex.

Let us come to the proof of Theorems 5.10 and 5.11. Theorem 5.10 is a consequence of the following simple lemma (its proof we omit).

Lemma 5.13 *Let $w = a_{k_1} a_{k_2} \dots a_{k_l}$ be a nonzero word from generators of an automata algebra A. Then φ maps w into the matrix $\varphi(w)$, such that its element with indeces (i, j) is equal to*

$$\varphi(w)_{ij} = \sum \tilde{t}_{k_1, i\, v_1} \; \tilde{t}_{k_2, v_1 v_2} \dots \tilde{t}_{k_l, v_{l-1}\, j},$$

where the summation is taken over all paths $i, v_1, v_2, \dots, v_{l-1}, j$, which connect the verteces i and j, and, such that marks on edges in each of these paths coinside with the word w. (If there are no such paths, then the sum is equal to zero.)

In the case, when $G(A)$ is a determinate graph (i.e., there is no edges, which begin in the same vertex, and have the same mark), then there can be not more, than one summand in this sum. □

Theorem 5.10 is a consequence of Lemma 5.13, because images of nonzero words, with respect to the map φ, are linearly independent.

In the case of the map ψ, Lemma 5.13 can be reformulated as follows.

Lemma 5.14 *Let the graph $G(A)$ of an automata algebra A is determinate (i.e., there is no edges, which begin in the same vertex, and have the same mark). Let $w = a_{k_1} a_{k_2} \dots a_{k_l}$ be a nonzero word from generators of A. If in $G(A)$ there exists a path from the vertex i to the vertex j, such that marks on its edges coincide with w, then the (i, j)-th element of the matrix $\psi(w)$ is equal to*

$$\psi(w)_{ij} = t_{k_1, iv_1}^{\alpha_{i\, v_1}} t_{k_2, v_1 v_2}^{\alpha_{v_1 v_2}} \dots t_{k_l, v_r\, j}^{\alpha_{v_r\, j}},$$

where by $v_1, v_2, \dots, v_r$ are denoted different verteces, through which the path passes, by $\alpha_{v_h v_t}$ is denoted the number of passings of the path through the edge $v_h \to v_t$.

In other words, the (i, j)-th element of the matrix is equal to the product of powers of commutative variables, which correspond to edges of the path from the *vertex i to the vertex j. The power of each this variable equals to the number of passings through this edge.*

If there is no such path (i.e., the path, which connects the vertex i with the vertex j, and, such that marks on its edges coincide with w), then the corresponding element of the matrix $\psi(w)$ equals zero. □

We shall need one more lemma to prove Theorem 5.11.

Lemma 5.15 *Let the determinate graph $G(A)$ of an automata algebra A doesn't have any verteces, which belong to two cycles. Then each path in $G(A)$ is uniquely defined by the number of passings of each edge.*

Proof. We shall use the induction on the length of a path. It is enough to prove that the first edge is defined uniquely. Let more than one edge of the path begin in the initial vertex (hence, the path passes through the initial vertex several times). Then only one of these edges can belong to a cycle, because the graph doesn't contain intersecting cycles. As the path has to return to the initial vertex, then the first edge of the path is a cyclic edge. □

Theorem 5.10 is a consequence of Lemmas 5.14 and 5.15, because, by them, we have that images of different nonzero words of A are linearly independent.

If a monomial algebra is representable, then its word semigroup is representable also. The inverse statement is wrong. The following result holds (see [45]).

Theorem 5.16 *Each regular language is representable by matrices. (In particular, each automata monomial semigroup is representable.)*

If a minimal graph contains linking cycles, then the corresponding monomial algebra is not representable. Hence, the kernel of the representation can contain not words, but only their linear combinations. To obtain the information about this kernel (in particular, the information about a T-ideal, which contains the kernel) is an interesting task. Also interesting is the problem, when the canonical representation of a graph is an exact representation of its word semigroup?

A monomial semigroup will be called quasirepresentable, if it has a representation without a monomial kernel. It is possible to describe semigroups, quasirepresentable over finite rings.

Proposition 5.17 *a) A semigroup is representable over a finite ring, only if it is finite.*

b) A monomial semigroup is quasirepresentable over some finite ring, only if it is automata.

Proof. The item a) is obvious. The quasirepresentability of a monomial semigroup can be deduced from the canonical representation. The inverse statement is a consequence of the language regularity criterion: a language is regular, only if all its words can be divided on a finite number of types in way, such that the substitution of any subword by a word of the same type preserves the membership to the language. □

5.2.2 Polynomial identities

Corollary 5.12 gives us an alogorithm of checking the existence of a polynomial identity in an automata algebra. The aim of this section is to obtain a more exact information about identities. To make the exposition simplier, we shall consider only the case of the infinite ground field.

Let us introduce some definitions and notations, which will be used in what follows (in particular, in Chapters 6 and 7).

Definitions. A cycle will be called irreducible, if its *big period*, i.e., the word, which can be read from it, is non-cyclic, i.e., this word is not a power of a smaller word. The *small period* of a cycle is the root of its big period. By a *position of a word in a graph* we shall call a path, from which the given word can be read. If there is a relation between graphs Γ' and Γ, which is one to one on arrows and verteces, and, such that to different marks on edges of one graph correspond different marks on edges of another, then we shall say that the graph Γ' differs from the graph Γ by the *letter sticking.*

Let us note that, if a word doesn't have any position in a graph, which define an automata algebra, then this word is a zero word in this algebra. All words, which correspond to a passing through some cycle are cyclically conjugate. All words, which positions connect two given verteces A and B of a cycle, are of the form R^kS, where R corresponds to the big period of the cycle and S – to the shortest path, which connect A and B.

Let us denote by A_u a monomial algebra, such that all its nonzero words are subwords of an infinite cyclic word $u^\infty = uuu\ldots$. Let us assume that u is noncyclic, i.e., u doesn't equal to a power of any of its proper subwords.

Theorem 5.18 *The set of polynomial identities of an algebra A_u coincides with the set of identities of the matrix algebra K_n, where n is the length of u.*

Proof. We can represent A_u by a cyclic graph $G(A)$ (we allow any vertex of the graph to be initial). Marks on edges of the cycle in the order of its passage coincide with u. Let us denote by $T(A)$ the ideal of A identities, and by $T(K_n)$ – the ideal of identities of the matrix algebra K_n. The construction of the representation ψ gives us the inclusion of A into the matrix algebra of order n over some extension F of the ground field K. As K is ininite, then an extension preserves the set of identities. Hence, $T(A) \supseteq T(F_n) = T(K_n)$.

The inverse inclusion is a consequence of those fact that the image $\psi(A)$ generates over F all algebra F_n. It is enough to check that matrix units belong to $F \cdot \psi(A)$. Let $u = a_{k_1} a_{k_2} \dots a_{k_n}$, let edges $i \to i+1$ of $G(A)$ be marked by letters a_{k_i}, $i = 1, \dots, n-1$, and let the edge $n \to 1$ be marked by a_{k_n}. By Lemma 5.15, we have that the image of the word

$$\psi(a_{k_i} a_{k_{i+1}} \dots a_{k_n} u a_{k_1} \dots a_{k_{j-1}})$$

is the matrix, such that all its elements, except (i, j)-th, are zeroes, and the (i, j)-th element is a nonzero element of F. To obtain the unit matrix e_{ij}, it is enough to multiply the matrix by the inverse element. $\square$

Let us remind that the polynomial degree or the complexity of a PI-algebra A is the maximal n, such that $T(A) \subseteq T(K_n)$.

Theorem 5.19 *The polynomial degree of an automata algebra equals to the maximum of length of periods of all infinite cyclic words in the algebra.* $\square$

Corollary 5.20 *The polynomial degree of an automata algebra is not greater, than the order of a maximal cycle in the algebra minimal determinate graph.* $\square$

The proof of the theorem we leave to the reader. Let us note that the inequality in the corollary condition can be strict, as the following example demonstrates:

$$\bullet \xrightarrow{a} \bullet \underset{b}{\overset{b}{\rightleftarrows}} \bullet \xrightarrow{c} \bullet$$

The maximal length of a cycle here is equal to two, but the polynomial degree is equal to one, because the identity $[x_1, y_1][x_2, y_2][x_3, y_3] = 0$ holds in this algebra.

Therefore, to find the complexity of an automata algebra, we have to compute the maximal length of small periods of all cycles.

The following proposition is a consequence of the independence theorem 2.38.

Proposition 5.21 *a) The minimal dimension of a non-nilpotent representation of a monomial algebra (not necessary automata or PI) is equal to the minimal length of the period of an infinite word in this algebra.*

b) In the case of an automata algebra, this dimension is equal to the minimal length of a small period in each graph, which define the algebra. In the case of a PI-algebra this dimension is equal to the algebra complexity. $\square$

In the end of this section we shall make one last commentary.

Proposition 5.22 *A commutative monomial algebra is a direct sum of* 1-*generated algebras.* $\square$

5.3 The structure theory of automata algebras

Automata algebras are like finite-dimensional algebras with respect to their structure. In particular, the Jacobson radical of an automata algebra is nilpotent and coincides with the intersection of a finite number of prime ideals.

Let us denote by $\Gamma(A)$ the minimal determinate graph of A. Let n be the number of verteces in $\Gamma(A)$.

Theorem 5.23 *The Jacobson radical, the Baire radical and the nilradical of an automata algebra coincide and are nilpotent. The radical, as a linear space, is generated by words, which correspond to those paths in $\Gamma(A)$, which don't belong to any cyclic path. The radical, as an ideal, is generated by such words of length $\leq n$.*

Let us remind that in an arbitrary monomial algebra the Jacobson radical coincides with the nilradical (Chapter 3), but the Baire radical (the prime radical) can be strictly less, than the Jacobson radical.

To prove the theorem, we shall need some simple statements.

Proposition 5.24 *Let A be an arbitrary monomial algebra, then*

(1) A is semiprime iff for each word $u \neq 0$ there exists a word v, such that $uvu \neq 0$;

(2) A is prime iff for each two words $u, v \neq 0$, there exists a word w, such that $uwv \neq 0$.

Proof. The necessity is obvious. The study of leading words in the corresponding elements gives us the sufficiency. □

Proposition 5.25 *Let us consider an arbitrary subgraph $G \subseteq \Gamma(A)$. Then the automata algebra A', which corresponds to G, is a homomorphic image of A.*

Proof. Indeed, the set of nonzero words of A' is a subset in the set of nonzero words of A. Hence, A' is a factor of A by the ideal, which is generated by the difference of these sets. □

Proposition 5.26 *Let an automata algebra A is semiprime, then it is semisimple in Jacobson sense.*

Proof. Let us consider an arbitrary element $0 \neq x \in A$. Let us denote by u the leading word of x. By Proposition 5.24, there exists a word v_1, such that $uv_1u \neq 0$. By applying again Proposition 5.24 to the word uv_1u, we can find a word v_2, such that $uv_1uv_2uv_1u \neq 0$, and so on. In result we shall obtain a right infinite word W, with infinite occurence of u. Let us mark the infinite number of non-overlapping occurences u in W: let u is an end of words $(W)_{k_1}, (W)_{k_2}, \ldots$, where by $(W)_i$ we denote a beginning of W of length i. As A is automata, then we can find two words in the set $(W)_{k_i}$, which correspond to the same vertex of the graph $\Gamma(A)$. Let these two words be $(W)_h$ and $(W)_t$, where $t > h$, $t - h > |u|$. The word $(W)_t$ can be represented, as $(W)_h zu$, where z is nonzero. As $(W)_h$ and $(W)_h zu$ are equivalent, then the infinite periodic word $U = (zu)^\infty = zuzu\ldots$ is nonzero in A. Let us consider the algebra A_U. In Chapter 3 it was proved that this algebra is semisimple in Jacobson sense. Under the natural epimorphism $A \to A_U$, the element x is mapped to a nonzero element in A_U (because the image of x is a linear combination of words with u as a term). Hence, $x \notin J(A)$. □

Remark. If A is an arbitrary monomial algebra, then Proposition 5.26 is wrong.

Proof of Theorem 5.23. Let us denote by I the ideal, which is generated by all words, such that corresponding paths are not contained in any cyclic path. Obviously, these words generate a nilpotent semigroup, hence, I is nilpotent. Now it is enough to prove that $J(A/I) = 0$.

Let us prove that the algebra A/I is semiprime. Indeed, A/I is an automata algebra, defined by the subgraph of the graph $\Gamma(A)$, such that its verteces and edges belong to cyclic paths (see Proposition 5.25). Hence, each nonzero word in A/I is contained in an infinite periodic word, therefore, by Proposition 5.24, A/I is semiprime. The semisimplicity in Jacobson sense is a consequence of Proposition 5.26. □

An oriented graph will be called strongly connected, if for each two its verteces there exists a path, which connects them.

Theorem 5.27 *The Jacobson radical of an automata algebra is equal to the intersection of a finite number of prime ideals P_i, which correspond to strongly connected components G_i of the graph $\Gamma(A)$. The ideal P_i is generated, as a linear space, by words, which correspond to those paths, which are not contained in G_i. As an ideal, P_i is generated by such words of length $\leq n$. The quotient algebra A/P_i is an automata algebra, which is defined by the graph G_i.*

Proof. Each cyclic path belong to some G_i. Therefore, the intersection of P_i is generated by words, such that the corresponding paths don't belong to any cyclic path. Hence, by Theorem 5.23, the intersection of P_i coincides with the radical. The primarity of P_i is a consequence of Proposition 5.24 and the strong connectivity of G_i. □

5.4 Nilpotent elements and zero divisors

The subject of this section is the construction of algorithms for checking the nilpotency and the zero divisibility of the given element x of an automata algebra A. We shall assume that A is defined by its graph $\Gamma(A)$.

Theorem 5.28 *Let the graph $\Gamma(A)$ of an automata algebra A has n verteces and $x \in A$ is a nilpotent element (i.e., there exisrs t, such that $x^t = 0$), then $x^n = 0$.*

Corollary 5.29 *The checking of the nilpotency of an arbitrary element $x \in A$ is an algorithmically solvable problem.* $\square$

Proof of the theorem. As it was proved above, A can be embedded in the algebra of $n \times n$ matrices over a free algebra (Theorem 5.10). It is well known that a free algebra can be embedded in a division algebra (see [72]). Hence, it is enough to prove the statement for a nilpotent matrix M over a division algebra. But in this case the theorem is a consequence of the dimension reasoning. (The left multiplication by a matrix is a linear operator on the right vector space of n-columns over the division algebra. The dimensions of images of M, M^2 and so on, strictly decrease, hence, $M^n = 0$.) $\square$

The problem about the zero divisibility is more difficult. We can attempt to manage it by the embedding into the matrix algebra over a free algebra, because the corresponding problem for this algebra is algorithmically solvable.

Proposition 5.30 *There exists an algorithm, which checks is an arbitrary $n \times n$ matrix a right zero divisor in the matrix algebra over a free algebra.*

The idea of proof. It is enough to check, that some nontrivial linear combination of the matrix rows is zero. The algorithm is similar to the Gauss method of reducing a matrix to the step-form. By elementary transformations of rows, we can achieve the situation, when the leading words of nonzero polynomials in the first column are not ends of each other. Hence, they constitute a free base of the left ideal, generated by them. After permuting the rows, we can assume that nonzero elements of the first column are positioned in rows $1, 2, \ldots, i_1$, and other elements in the first column are zeroes. This process is repeated, as in the Gauss method, for the next minor, which situated in rows $i_1 + 1, \ldots, n$ and columns $2, \ldots, n$, and so on. If in the end we shall obtain a matrix with a zero row, then the initial matrix is a right zero divisor (because the elementary transformations correspond to the left multiplication by invertible matrices). If the final matrix doesn't contain a zero row, then each linear combination of rows of this matrix is nonzero, hence, it cannot be a right zero divisor. As the initial matrix can be obtained as a product by an invertible matrix, then the same is true for the initial matrix also. $\square$

Remark. By improving the proof, we can obtain that, if a matrix M is a right zero divisor, then

1) the left annihilator of M is generated by an idempotent;

2) there exists an annihilating matrix L, such that $LM = 0$ and degrees of elements of L are not greater, than $dn(n+1)/2$, where d is the maximal degree of elements of M.

Unfortunately, it is not possible to use Proposition 5.30 for checking, is an element $y \in A$ a zero divisor, or not (A is an automata algebra). By embedding

A into the matrix algebra, we can prove that y is not a zero divisor, if the corresponding matrix is not a zero divisor. But in the case of the inverse statement, it is not clear, has the left annihilator of the image of y a nonzero intersection with the image of A, or not. Therefore, we have to use more complex reasoning.

5.4.1 An algorithm for checking, is an arbitrary element a zero divisor or not?

By $a_1, \ldots, a_s$ let us denote generators of an automata algebra A. Let us fix an element $y \in A$. Our task is to check, is this element y a right zero divisor, i.e., does there exist an element $x \in A$, such that $xy = 0$? Let $\deg y = n$, n will be fixed in what follows. Let the minimal graph $\Gamma(A)$ has m verteces. Let us remind that the verteces of $\Gamma(A)$ are in one to one relation with the equivalency classes of nonzero words in A. Two words u and v are equivalent, if their sets of nonzero extension coincide:

$$u \sim v \iff (\forall w \quad uw \neq 0 \iff vw \neq 0) \ .$$

An algebra is automata, if $|\Gamma(A)| < \infty$, i.e., if there exist m nonzero words $d_1, \ldots, d_m$, such that each nonzero word from A is equivalent to one of d_i. Let us note that, if u and v are equivalent, then the set of elements (not necessary words) x, such that $ux = 0$, coincides with the set of elements z, such that $vz = 0$. (This statement is a consequence of those fact that in a monomial algebra different words cannot be cancelled.)

Let us prove at first the auxiliary lemma. We assume that A is an algebra over the field K and $1 \in A$.

Lemma 5.31 *Let there exists an element $x \in A$, such that $xy = 0$. Then there exist a word d and an element x', such that $dx'y = 0$ and the free term of x' is nonzero.*

Proof. Let the free term of x equals zero. Then $x = a_1x_1 + \cdots + a_sx_s$, where a_i are generators of A (i.e., letters) and the degree of elements x_i is smaller, by 1, than the degree of x. As words, which begin with different letters, cannot cancel, then $a_ix_iy = 0$, for all i. Let us take i_1, such that $x_{i_1} \neq 0$. If the free term of x_{i_1} is nonzero, then we can take a_{i_1}, as d. Otherwise, we can perform the same procedure with x_{i_1}, i.e., to find a letter a_{i_2}, such that $a_{i_1}a_{i_2}x_{i_1,i_2}y = 0$, and so on. This process will stop at some moment. $\square$

Let us remind that our aim is to find x, such that $xy = 0$ (y is fixed and its degree equals n). By $\bar{d}$ will be denoted the equivalency class of a nonzero word $d \in A$, i.e., the vertex of the minimal graph $\Gamma(A)$. In the algorithm construction the main role will be played by the following linear subspaces in A:

$$\begin{aligned} W_{\bar{d},k} &= \{x\colon \deg x \leq k \quad \& \quad dxy = 0\}, \\ V_{\bar{d},k} &= \{x\colon \deg x \leq k \quad \& \quad \deg(dxy) < \deg d + n\}. \end{aligned}$$

Commentary: to solve our problem it is enough either to find a nonzero element $x \in W_{\bar{d},k}$ for some word d and degree k, or to prove that spaces $W_{\bar{d},k}$ are zero, for all d and k. It is more convenient to work with spaces $V_{\bar{d},k}$. In their definition the condition $dxy = 0$ is substituted for a slightly weaker one: in the linear combination, which represents dxy, all terms with degrees $\geq \deg d + \deg y$ are canceled. Let us note that $W_{\bar{d},k} \subseteq V_{\bar{d},k}$. Moreover, the subspace $W_{\bar{d},k}$ in $V_{\bar{d},k}$ is defined by a finite number of linear conditions on conponents with degrees $\leq n$.

The following lemma is a direct consequence of definitions.

Lemma 5.32 *Subspaces $V_{\bar{\cdot},k}$ are embedded into each other and increase, when k increases: $V_{\bar{d},1} \subseteq V_{\bar{d},2} \subseteq V_{\bar{d},3} \subseteq \dots$. Analogously, $W_{\bar{d},1} \subseteq W_{\bar{d},2} \subseteq W_{\bar{d},3} \subseteq \dots$.* □

Let us consider the natural epimorphism $\pi_n : A \to A/A^{(n+1)}$ of the algebra A into its quotient algebra by the ideal, generated by all words of degree $> n$. We shall study the images of $W_{\bar{d},k}$ and $V_{\bar{d},k}$ under this epimorphism:

$$V_{\bar{d},k}^{(n)} = \pi_n(V_{\bar{d},k}) \ ,$$
$$W_{\bar{d},k}^{(n)} = \pi_n(W_{\bar{d},k}) \ .$$

The space $V_{\bar{d},k}^{(n)}$ is the projection of $V_{\bar{d},k}$ into the subspace, generated by all words of degree $\leq n$. As this spaces are finite-dimensional and increase, when k increases (Lemma 5.32), then, for some $k = N$, the stabilization begins: for each $k \geq N$ and for each $\bar{d} \in \Gamma(A)$, the equality $V_{\bar{d},k}^{(n)} = V_{\bar{d},N}^{(n)}$ holds. The base of our algorithm is the following proposition.

Proposition 5.33 *Let for some natural k and for all verteces $\bar{d}$ in the minimal graph $\Gamma(A)$ of an algebra A, the equality $V_{\bar{d},k-1}^{(n)} = V_{\bar{d},k}^{(n)}$ holds. Then the equality $V_{\bar{d},k}^{(n)} = V_{\bar{d},k+1}^{(n)}$ holds, for all $\bar{d}$.*

This propositions states that, if stabilization occurs at some step k, then the subspaces $V_{\bar{d},k}^{(n)}$ will not increase.

Proof. Let us fix $\bar{d} \in \Gamma(A)$ (d is a word). Let $\bar{x} \in V_{\bar{d},k+1}^{(n)}$. We have to prove that $\bar{x} \in V_{\bar{d},k}^{(n)}$. By the definition, there exists an element $x \in V_{\bar{d},k}$, such that $\pi_n(x) = \bar{x}$. Let us represent x, as $x = x_0 + a_1x_1 + \dots + a_sx_s$, where x_0 is a free term (i.e., an element of the ground field) and $a_1, \dots, a_s$ are A generators. The fact that $x \in V_{\bar{d},k+1}$ means that $\deg(dxy) < \deg d + n$. We have

$$dxy = dx_0y + \sum_{i=1}^{s} da_ix_iy.$$

Let us consider terms in the right hand part of the equality with degrees $\geq \deg d + n + 1$. The degree of dx_0y is $\leq \deg d + n$, hence, terms with degrees $> \deg d{+}n$ cannot be canceled with terms in dx_0y. Therefore, $\deg(\sum_i da_ix_iy) < \deg d + n + 1$. Terms, which begin with different subwords da_i, cannot be canceled, because the letters a_i are different for different i, hence $\deg(da_ix_iy_i < \deg d + n + 1$. But, this exactly means that $x_i \in V_{\overline{da_i},k}$. Let us consider now terms of degree exactly $\deg d + n$. They depend only on those terms of the element x, which have degree $\leq n$. The fact that all of them are zero means that $\deg(\pi_{\deg d+n}(dxy)) < \deg d + n$. So, we have

$$x \in V_{\bar{d},k+1} \iff \begin{array}{l} \forall i \quad da_ix_i \in V_{\overline{da_i},\,k} \quad \& \\ \deg(\pi_{\deg d+n}(dxy)) < \deg d + n. \end{array} \tag{5}$$

By the equality $V_{\overline{da_i},k} = V_{\overline{da_i},k-1}$, we have that, for each $i = 1, \ldots, s$, there exists an element $x_i' \in V_{\overline{da_i},k-1}$, such that $\pi_n(x_i') = \pi_n(x_i)$. Let us construct the element x' from them:

$$x' = x_0 + \sum_{i=1}^{s} a_ix_i' \ .$$

The degree of x' is $\leq k$. The condition $\deg(\pi_{\deg d+n}(dx'y)) < \deg d + n$ is a consequence of those fact that the terms of x and x' of degrees $\leq n$ coincide. Therefore, by the equality (5), where k is substituted by $k-1$, we have that $x' \in V_{\bar{d},k}$. By the construction of x', $\pi_n(x') = \pi_n(x)$ and $\pi_n(x') \in V^{(n)}_{\bar{d},k}$. Hence, $\bar{x} = \pi_n(x) = \pi_n(x') \in V^{(n)}_{\bar{d},k}$. □

Proposition 5.34 *Let for some natural k and for all verteces $\bar{d}$ in the minimal graph $\Gamma(A)$ of an algebra A the equalities $V^{(n)}_{\bar{d},k+1} = V^{(n)}_{\bar{d},k}$ hold. Then the equalities $W^{(n)}_{\bar{d},k+1} = W^{(n)}_{\bar{d},k}$ hold also, for all $\bar{d}$.*

Proof. Let $\bar{x} \in W^{(n)}_{\bar{d},k+1}$, i.e., $\bar{x} = \pi_n(x)$ and $x \in W_{\bar{d},k+1}$. The last condition means that $\deg x \leq k+1$ and $dxy = 0$. The equality $dxy = 0$ is equivalent to the following conditions:

$$\begin{array}{l} \text{a) } \deg(dxy) < \deg d + n \text{ and} \\ \text{b) } \pi_{\deg d+n}(dxy) = 0, \text{ i.e, terms with degrees} \leq \\ \deg d + n \text{ are equal to zero.} \end{array} \tag{6}$$

The first condition means that $d \in V_{\bar{d},k+1}$, the second is defined by a finite number of linear equations on coefficients of terms with degrees $\leq n$. As $V^{(n)}_{\bar{d},k+1} = V^{(n)}_{\bar{d},k}$, then there exists an element $x' \in V_{\bar{d},k}$, such that $\pi_n(x') = \pi_n(x)$. Then a) (6) holds, by the definition of $V_{\bar{d},k}$, and b) (6) holds, because terms with degrees $\leq n$ in x and x' coincide. Therefore, by (6), $dx'y = 0$. The degree of x' is $\leq k$, hence, $x' \in W_{\bar{d},k}$ and $\pi_n(x') \in W^{(n)}_{\bar{d},k}$. So, $\bar{x} = \pi_n(x) = \pi_n(x') \in W^{(n)}_{\bar{d},k}$. The proposition is proved. □

Lemma 5.35 *If, for all verteces $\bar{d}_i \in \Gamma(A)$ and for all k, the spaces $W^{(n)}_{\bar{d}_i,k}$ are zero, then y is not a right zero divisor.*

Proof. Let the contrary is true, then, by Lemma 5.31, there exist a word d_l and an element x' with a nonzero free term, such that $d_l x' y = 0$. By the definition of $W_{\bar{d}_l,k}$, $x' \in W_{\bar{d}_l,\deg x'}$. But $\pi_n(x') \neq 0$, because x' has a nonzero free term. Hence, $W^{(n)}_{\bar{d}_l,\deg x'} = \pi_n(W_{\bar{d}_l,\deg x'}) \neq 0$. We have a contradiction. □

The construction of the algorithm. For all verteces $\bar{d}_i$, $i = 1, \ldots, m$, of the minimal graph $\Gamma(A)$ of an algebra A we compute the spaces $V_{\bar{d}_i,k}$ $(k = 1, 2, \ldots)$. For each k we compute the projections $V^{(n)}_{\bar{d}_i,k}$ of these spaces in the quotient algebra $A/A^{(n+1)}$, by the ideal, generated by all words of degrees $> n$. These projections are finite-dimensional and their dimensions are bounded from above by some number, which is independent from k (i.e., by the maximal number of all paths in $\Gamma(A)$ of length $\leq n$, which begins in same vertex). By Lemma 5.32, the spaces $V^{(n)}_{bard,k}$ increase, when k increases. Let for $k = N$ the stabilization begins: for all d_i, the equalities $V^{(n)}_{\bar{d}_i,k} = V^{(n)}_{\bar{d}_i,k-1}$ hold. By Proposition 5.33, the growth of $V^{(n)}_{\bar{d},k}$ stops at that moment. By Proposition 5.35, the growth of $W^{(n)}_{\bar{d},k}$ also stops at that moment. Let us find the spaces $W_{\bar{d}_i,N)}$, for all verteces $\bar{d}_i$. If one of them is nonzero, for example $W_{\bar{d}_l,N)} \neq 0$, then there exists a nonzero element x, such that $d_l x y = 0$, i.e, $d_l x$ is the required zero divisor. Otherwise, all spaces $W(n)_{\bar{d}_i,k}$ are zero and, by Lemma 5.35, y is not a right zero divisor.

The complexity of the algorithm. The constructed above algorithm has "a polynomial complexity modulo growth of algebra", i.e., if the algebra growth is polynomial, then our algorithm is also polynomial (with respect to the y degree). If the algebra growth is exponential, then our algorithm is exponential also. The algorithm has a quadratic dependence on the number of the graph verteces.

Let $n = \deg y$, $m = |\Gamma(A)|$ and $r(n)$ be the growth function of the algebra (i.e., the number of words of length $\leq n$).

It is easy to obtain an estimation on the degree of the y annihilator. The sum of dimensions of the spaces $V^{(n)}_{\bar{d}_i,k}$ is not greater, than $m \cdot r(n)$, hence, the stabilization will begin not later, than after the defined above number of steps. Let z be a nonzero element of the minimal degree in the y annihilator, i.e., $zy = 0$. Then $z = dx$, where $x \in W_{\bar{d}_i,k}$ and $k \leq m \cdot r(n)$. We can choose the element d in a way, such that its degree is not greater, than the number of verteces in $\Gamma(A)$ (an element of such degree can be found in each equivalency class of words in A). So, we have $\deg d \leq m$, $\deg x \leq m \cdot r(n)$. Hence, $\deg x \leq m \cdot r(n) + m$.

5.5 The Noetherity

Proposition 5.36 *An automata algebra is right Noetherian iff*

1) its minimal determinate graph $\Gamma(A)$ *doesn't have any twice cyclic verteces and*

2) if $\Gamma(A)$ *contains cycles, then not one of them contains an outgoing edge.* □

Corollary 5.37 *A Noetherian automata algebra is a PI-algebra and either it has a linear growth, or it is finite-dimensional.* □

The proof is obvious. However, a more strong statement is valid.

Theorem 5.38 *A monomial algebra A is right Noetherian, only when its graph (tree) of right multiplications contains only a finite number of branchings.*

The proof of this statement see in the book by J.Okninski [87].

The following statement is a consequence of this theorem and the periodicity theorem.

Corollary 5.39 *A right Noetherian monomial algebra is automata.* □

And, by Corollaries 5.39 and 5.37, we have

Corollary 5.40 *A right Noetherian monomial algebra is a PI-algebra and either has a linear growth, or is finite-dimensional.* □

6 Representations of monomial algebras

This chapter is dedicated to representations of monomial algebras. We consider "tame" and "wild" monomial algebras, i.e., algebras, which representations can or cannot be classified. Only 1- and 2-generated algebras with zero multiplication are tame. The description of irreducible representation of a monomial algebra (if its definition is "good") can be reduced to the description of representations of an algebra A_u. All nonzero words in A_u are subwords of the infinite cyclic word u^∞. If the minimal graph of an algebra contains linking cycles (i.e., the algebra has an exponential growth), then the classification problem about its irreducible representations is wild.

The necessary condition of the representability is the validness of the height theorem. In this case, there exists a number h, such that all words in the algebra are of the form $u_1^{k_1}u_2^{k_2}\dots u_l^{k_l}$, where $l \le h$ and $\{u_i\}$ is some fixed set of words. The necessary and sufficient condition of the representability can be formulated as a condition on the set of power vectors $\vec{k} = \langle k_1, \dots, k_l\rangle$: the set of $\langle k_1, \dots, k_l rangle$, such that $u_1^{k_1}\dots u_l^{k_l} = 0$, is defined by a system of exponential Diophantine equations.

Then we study varieties, generated by monomial algebras: $M_n = \mathrm{Var}(A_u)$ and the variety, generated by upper triangular matrices,for example. We shall need these varieties for the combinatorial study of the identities complexity.

To study all these problems we need the technique, which is related to the graph definition of representations, to representations of direct sums and tensor products.

6.1 The classification of representations: wild and tame problems

An algebra is called "tame", if all its representations can be classified, otherwise, it is called "wild".

Theorem 6.1 *The following algebras are tame:*
1) 1-generated algebras,
2) 2-generated algebras with the zero multiplication.
All other algebras are wild.

Proof. The case 1) is obvious. Let V be a representation space of a 2-generated algebra A with generators a and b and with the zero multiplication. Then to an A representation corresponds the pair of operators $\tilde{A}$ and $\tilde{B}{:}V/(ker\quad A \cap ker\quad B) \to ker\quad A \cap ker\quad B$. And to each pair of operators $M \overset{a}{\underset{b}{\rightrightarrows}} N$ can be related a representation of such algebra. Hence, the problem of classification of such algebra representations can be reduced to the classical problem of the linear algebra – the classification of pairs of operators $U \overset{a}{\underset{b}{\rightrightarrows}} V$, which solution is known [10].

Analogously, the classification of representations of a 3-generated monomial algebra with the zero multiplication can be reduced to the problem of classification of triples of operators $U \overset{\to}{\underset{\to}{\to}} V$, which is wild.

So, it remains to prove that the problem of classification of representations of a 2-generated monomial algebra with a nonzero multiplication is wild. It is enough to prove this fact in two following cases: the algebra A_1 with relations $a^2 = b^2 = ba = 0$ and the algebra A_2 with relations $a^3 = ab = ba = b^2 = 0$.

Let us consider the diagrams

$$\begin{tikzcd} V_3 \arrow[r, "a"] & V_4 \arrow[d, "b"] \\ V_1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & V_2 \end{tikzcd} \qquad \begin{tikzcd} V_3 \arrow[r, "a"] & V_4 \arrow[d, "a"] \\ V_1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & V_2 \end{tikzcd}$$

for the first algebra, for the second algebra.

Let the arrows, which connect V_1 and V_2, be isomorphisms, let $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ and let us define operators A and B using the diagrams: A is the direct sum of all operators, which correspond to arrows, marked by a.

We have: $V_2 = \operatorname{Im} B \cap \operatorname{Im} A$, $V_1 = A^{-1}(V_2) \cap B^{-1}(V_2)$. Положим $N = \operatorname{Im} AB$ Let $N = \operatorname{Im} AB$, in the first case, and $N = \operatorname{Im} A^2$, in the second, then $N \subseteq V_2$. So we come to the problem about the classification of pairs of operators-isomorphisms, which connect two different spaces and a subspace N in the second space. If we identify these spaces by the action of one of those operators, then we shall come to the problem about the classification of an operator $a^{-1}b : V_2 \to V_2$ and a subspace $N \subseteq V_2$. This problem is wild (see [10]).

6.2 Irreducible representations of monomial algebras

So, the problem about the classification of representations is wild, if an algebra is more or less interesting. Hence, we have to restrict the problem to the classification of irreducible representations. We shall assume in this section that the ground field F is algebraically closed.

Let us consider at first the case of an automata PI-algebra. The problem about the classification of irreducible representations can be reduced in usual way (by the factoring by the radical and by the decomposition of an operator algebra into a direct sum) to the prime case. Hence, we have to study irreducible representations over F of an algebra A_u, such that all its nonzero words are subwords in the infinite word u^∞.

Proposition 6.2 *Let t be a sum of words, which are cyclically conjugate to u. Then t generates the center of A_u: $Z(A_u) = F[t]$. The algebra A_u is a free module of dimension $V_{A_u}(|u|)-1$ over its center. (Let us remind that by $V_{A_u}(n)$ we denote the number of nonzero words of length $\leq n$ in A_u.)*

Proof. Obviously, t is central. The second part of the proposition is a consequence of those fact that the beginning (the end) of length n of each subword in u^∞ uniquely defines this subword (Proposition 2.3). $\square$

Let us consider a finite-dimensional representation φ of an algebra A_u, i.e, we consider a right module M over A_u, which is finite-dimensional, as a linear space over F. Let us denote by $T = \varphi(t)$ the operator of right multiplication by t. The following statement is obvious.

Proposition 6.3 *For each $\lambda \in F$ and $n \in \mathbb{N}$, the kernel and the image of the operator $(T - \lambda E)^n$ (E is the identical operator) are invariant subspaces. Hence, each representation decomposes into the direct sum of representations, which correspond to eigenvalues of T.* $\square$

Corollary 6.4 *If the representation φ is irreducible, then T is a dilation, $T \neq 0$.*

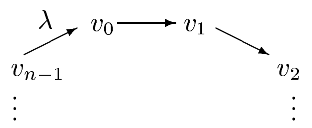
Proof. This corollary is a consequence of the previous proposition and Schur lemma. $\square$

Let us consider an irreducible representation φ of an algebra A_u. The operator $T = \varphi(t)$ is the operator of multiplication on some $\lambda \neq 0$. Let us consider the operator $\varphi(u)$. Obviously, $\varphi(u) \neq 0$ (otherwise $\varpi(t)^2 = 0$). Let us denote by v an eigenvector of $\varphi(u)$. As $ut = u^2$, then its eigenvalue equals λ, $vu = \lambda v$. Let $n = |u|$ and let us denote by $(u)_0, (u)_1, \ldots, (u)_{n-1}$ the beginning subwords of u of lengths $0, 1, \ldots, n-1$, respectively. By $u^{(i)}$ will be denoted the word, which is cyclically conjugate to u and which first letter has the i-the position in u, $i = 0, 1, \ldots, n-1$. Then $u^{(0)} = u$ and $t = \sum_{i=0}^{n-1} u^{(i)}$. Obviously,

$$u(u)_i u^{(i)} \neq 0, \quad \textit{and} \quad u(u)_i u^{(j)} = 0, \text{ if } i \neq j. \tag{7}$$

Let us consider the vectors $v_0 = v = v(u)_0, v_1 = v(u)_1, v_2 = v(u)_2, \ldots, v_{n-1} = v(u)_{n-1}$. It is easy to check that they are linearly independent and constitute a base in an invariant subspace. The linear independence is a consequence of (7) and of those fact that $v_i = (1/\lambda)vu(u)_i$. The invariancy is a consequence of the equality $vu = \lambda u$.

As our representation is irreducible, then the vectors v_i constitute a base in the representation space. It is easy to obtain matrices, which correspond to A generators. Let us consider the cycle



and let us mark the arrow $v_i \to v_{i+1}$ by the letter a_{k_i}, if the word $(u)_{i+1}$ ends with this letter. The arrow $v_{n-1} \to v_0$ we shall mark by the last letter of u (let us denote this letter by $a_{k_{n-1}}$). We have

$$\begin{array}{lcl} v_i a_{k_i} & = & v_{i+1}, \quad i = 0, \ldots, n-2, \\ v_{n-1} a_{k_{n-1}} & = & \lambda v_0. \end{array}$$

So, we defined linear operators, which correspond to the generators of the algebra A.

Remark 1. The represented above cycle corresponds to the graph of A, one of which arrows is additionally marked by λ. The choice of another arrow corresponds to the dilation of base vectors by λ and by λ^{-1}.

So, we have a description of irreducible representations of an algebra A_u: each irreducible representation is uniquely defined by a constant $0 \neq \lambda \in F$.

Remark 2. The problem about the classification of irreducible representations of an automata algebra, which is not a PI-algebra, is wild: it contains the problem about the classification of representations of a free 2-generated algebra or the problem about the classification of pairs of operators-isomorphisms.

6.3 Some constructions

6.3.1 Operations over monomial algebras. Direct sums

Proposition 6.5 *Let A be a monomial algebra, $a_{11}, \dots, a_{1k_1}$, $a_{21}, \dots, a_{2k_2}, \dots,$ a_{sk_s} be its generators. Let $b_i = \sum_{j=1}^{k_i} a_{ij}$, then*

a) elements $b_1, \dots, b_s$ generate a monomial algebra;

b) a word $U(b_1, \dots, b_s) = 0$, only when, for each substitution $b_i \to a_{ij}$ (different occurences of b_i can be substituted by a_{ij} with different indeces j), its result (the value of U) is zero.

Proof. If we substitute b_i by the sum $\sum_j a_{ij}$ and remove the parentheses, then terms, which correspond to different words don't coincide. Therefore, the elements b_i generate a monomial algebra. The item b) is a consequence of those fact that terms, which appear after the removing of parentheses, are different and are in one to one relation with the defined above substitutions. $\square$

The following statement is a direct consequence of this proposition.

Proposition 6.6 (on the diagonal embedding) *Let $A_j = \langle a_{j1}, \dots, a_{js} \rangle$ be monomial algebras and let $A = \oplus A_j$. If $a_i = \sum_j a_{ji}$, then the elements $a_1, \dots, a_s$ generate a monomial algebra $\widehat{A}$ and the set of zero words in $\widehat{A}$ is the intersection of sets of nonzero words in A_j.* $\square$

Corollary 6.7 *Let A be a monomial algebra, $I_1, \dots, I_n$ be monomial ideals. Then, if the algebra A/I_j is representable, for each j, then the algebra $A/\cap_j I_j$ is also representable.* $\square$

Let Γ be the graph of a monomial algebra A. Let us mark Γ arrows by different letters and let these letters be the generators of the new algebra A_Γ. We can define an obvious multiplication in A_Γ: $b_1 b_2 = 0$, if the end of the arrow b_1 doesn't coincide with the beginning of the arrow b_2, otherwise, we have a nonsero monomial $b_1 b_2$. Then the correspondence $a_i \to \sum b_i$ (b_i correspond to arrows, marked by a_i) defines an embedding of the monomial algebra A into A_Γ and A_Γ will be called a Γ-cover of A.

Proposition 6.8 *a) Let A be a monomial algebra, $\widehat{A}$ be its Γ-cover. Then to each word in A corresponds the sum of words in A_Γ, where terms in the sum correspond to different positions of the given word in Γ.*

b) If $\Gamma(A)$ is produced from $\Gamma(A')$ by sticking of letters, then A can be embedded in A' and $\mathrm{Var}(A') \subseteq \mathrm{Var}(A)$. $\square$

Proposition 6.9 *a) If each nonzero word in A' is also nonzero in A, then A' is a quotient algebra of A and* $\mathrm{Var}(A') \subseteq \mathrm{Var}(A)$. *The factorization corresponds to the erasing of arrows in the graph.*

b) Let Γ_i be connected components of Γ. Then there exists a natural embedding of A_Γ into the direct sum of A_{Γ_i}. The variety $\mathrm{Var}(A_\Gamma)$ *is the union of* $\mathrm{Var}(A_{\Gamma_i})$. □

(By a position of a word we call a path, which edges, in the order of their passage, are marked by letters of the given word. If a word doesn't have any position, then this is a zero word. The same is true for superwords also.)

The next construction is related to tensor products. The tensor product is not defined in the category of monomial algebras (for example, $k\langle x\rangle \otimes k\langle y\rangle \simeq k[x, y]$, but the ring of commutative polynomials from two variables is not a monomial algebra). Hence, we have to choose an appropriate subalgebra in the tensor product.

Proposition 6.10 *Let A_i, $i = 1, \dot{;}n$, be a family of monomial algebras with generators a_{ij}, $j = 1, \ldots, s$. Let us consider the subalgebra $\widehat{A}$ in the tensor product $\otimes_i A_i$, which is generated by elements $a_{1j} \otimes a_{2j} \otimes \cdots \otimes a_{nj}$, $j = 1, \ldots, s$. Then $\widehat{A}$ is a monomial algebra. A word in $\widehat{A}$ is a zero word, only when it equals zero in one of A_i.* □

Hence, the ideal of words in $\widehat{A}$ is the sum of ideals of words in algebras A_i.

If all A_i are representable, then, considering the tensor product of representations, we have

Proposition 6.11 *a) Let as consider the algebra $\widehat{A}$ and the family A_i from the previous proposition. Let A_i are representable and let W_i be the corresponding spaces of representations. Then $\widehat{A}$ is also representable and its space of representation is $\otimes W_i$.*

b) Let A be a monomial algebra, $I_1, \ldots, I_k$ be monomial ideals (i.e., ideals, generated by sets of monomials) and the algebras A/I_j be representable, then the algebra $A/(I_1 + \cdots + I_k)$ is also representable. □

So, we can use operations of the intersection, of sum and of union, when we work with sets of nonzero words, such that factors by them are representable.

Further we shall consider only those graphs, which don't have any intersecting cycles. As usual, to different arrows will correspond different generators of a monomial algebra.

Proposition 6.12 *Let to different arrows in the graph correspond different generators. Then to the erasing of an arrow corresponds the factorization by the generator (by which this arrow was marked).* □

Definition 6.13 Two arrows in a graph are called parallel, if there doesn't exist a path, which contains both of these arrows.

The following proposition will be used in the reduction process.

Proposition 6.14 *If $\Gamma(A)$ contains parallel arrows, then A can be embedded in the direct sum of algebras A_α, where each A_α is constructed by the erasing of one of these arrows. The T-ideal of identities of A is the intersection of T-ideals in A_α, and the variety, generated by A is the union of corresponding varieties. Let I be a monomial ideal in A, such that the quotient algebra A_α/I is representable, for each α. Then the algebra A/I is also representable.* $\square$

In what follows we shall consider graphs without parallel arrows (and without linking cycles). Such graph is of the form

$$\to\bigcirc\to\bigcirc\to\cdots\to\bigcirc\to \; .$$

A graph of a nilpotent algebra without parallel words is a graph of the form $\to\to\to\to\to\to$, and the algebra itself is an algebra of subwords of a finite word.

Let us remind that an oriented graph is called strongly connected, if, for each two its verteces, there exist a path from the first vertex to the second. Obviously, in the PI-case all strongly connected components of a graph without parallel edges are cycles.

Proposition 6.15 *If a graph doesn't contain any parallel arrows, then each two verteces can be connected by a path. The quotient graph by strongly connected components is a linearly ordered set and the graph itself is of the form*

$$\to\to\bullet\to\to\bullet\to\cdots\bullet\to\to$$

where by black circles are denoted strongly connected components. $\square$

Proposition 6.16. Let the set of all subwords of a superword W constitutes a regular language. Then the graph of the algebra A_W doesn't have any parallel edges and is of the form $\bullet\to\to\to\bullet$. In the case of a right superword, the graph is of the form $\to\to\to\bullet$, in the case of a left superword – $\bullet\to\to\to$.

Proof. Each vertex has the next and the previous. Hence, there are no heads and no tails. As there are a finite number of stronly connected components and they are linearly ordered, then a begining of W (and an end of W) are in the same component. $\square$

Proposition 6.16 *If W is prime, then there is only one component and the graph is of the form $\bullet$.* $\square$

We shall consider quotient algebras of automata algebras, which are not automata themselfs. However, we shall use the graph technique all the same.

6.3.2 The semidirect product of monomial algebras

Definition 6.17 By the semidirect product of algebras A and B (denoted by $A \rtimes B$) will be called the quotient algebra of the algebra $A + A * B + B$, by the ideal, generated by elements of the form $b * a$, $b \in B$, $a \in A$.

The following result, due to J.Levine [83], is well known.

Theorem 6.18 *$T(A \rtimes B) = T(A) \cdot T(B)$, where $T(A)$ is the ideal of identities of A.* $\square$

Proposition 6.19 *The free product and the semidirect product are defined in the category of monomial algebras.* $\square$

Remark. We defined the semidirect product of algebras without unit. In the case of monomial algebras with unit, the definition must be changed in the obvious way.

We shall need the following proposition for the reduction process.

Proposition 6.20 *Let a graph Γ be of the form $\Gamma_1 \overset{\alpha}{\to} \Gamma_2$, The letter α is not occured in Γ_1 and each letter from Γ_2 is not occured in Γ_1. Let also there be a path from each vertex in Γ_1 to each vertex in Γ_2. Then the algebra of the graph Γ is isomorphic to the semidirect product of the algebra of Γ_1 and the algebra of the graph $\bullet \xrightarrow{\alpha} \Gamma_2$.*

Proof. Let a_2 be a letter, which marks some arrow in Γ_2, and A_{ij} be verteces in Γ_2, in which arrows, marked by a_i, begin. Let us consider verteces A_k in Γ_1 (except those, in which begins the arrow, which connectes Γ_1 and Γ_2) and all paths v_{ijk} from A_k to A_{ij}. Let

$$a_i' = a_i + \sum v_{ijk} a_i.$$

Obviously, a_i' generate a monomial algebra, which is isomorphic to the algebra of the graph $\bullet \xrightarrow{\alpha} \Gamma_2$, and the operation of the multiplication of the algebra of Γ_1 by this algebra, satisfies the properties of the semidirect product. Moreover, the algebra of Γ_1 and a_i' generate the algebra of Γ. $\square$

Corollary 6.21 *If a graph Γ doesn't have any parallel edges and all its arrows are marked by different letters, the the variety, generated by the algebra of Γ, is a semidirect product of matrix varieties and varieties, defined by identities of the form $x_1 x_2 \dots x_{n_i} = 0$. The corresponding T-ideal is a product of identity ideals of these algebras.* $\square$

Hence, we have the classification of varieties, generated by graphs, such that all their arrows are marked by different letters: the corresponding T-ideals can be produced by the union and by the intersection from T-ideals of varieties, defined by matrix algebras and the algebra with the identity $x_1 x_2 = 0$.

6.3.3 Morphisms and representations of algebras A_u

If a word u can be produced from a word v, by sticking the letters, then we have an embedding $A_v \to A_u$: $a \to \sum a_i$, where a_i are the letters, which occupy the same positions in u, as a occupies in v. This morphism corresponds to the morphism from Proposition 6.8. We shall be interested in other morphisms also.

Proposition 6.22 *Let u be a noncyclic word. Then all subwords in u^∞ of length k are lexicographically comparable and generate a monomial algebra. If $k \geq n = |u|$, then this algebra is a direct sum of $\gcd(k,n)$ subalgebras, each of them is isomorphic to A_v, where $|v| = n/\gcd(k,n)$ and all letters in v are different. (By $\gcd(k,n)$ we denote the greatest common divisor of k, n.)*

Proof. Positions of words of length k in the cycle of length n correspond to chords, which are equal to each other and which constitute $\gcd(k,n)$ same closed broken lines. These broken lines correspond to the mentioned above subalgebras. If $k \geq n$, then each word of length k has the unique position in the cycle (see Proposition 2.3) and all letters, which mark segments of all broken lines, are pairwise different. $\square$

If $k = n + 1$, then we get the embedding $A_v \to A_u$, where all letters in v are differernt. Using the previous reasoning, we have

Proposition 6.23 *Let u and v be arbitrary noncyclic words of the same length n. Then there exists an embedding $A_v \to A_u$, such that to each generator of A_u corresponds the sum of words of length $n+1$ in A_v.* $\square$

6.4 The criterion of the representability of a monomial algebra

If a monomial algebra is representable, then it is PI and in it the height theorems holds. The inverse statement is wrong. (For example, an algebra with a non-integral Gelfand-Kirillov dimension cannot be representable.) We shall formulate and prove the representability criterion.

By the height theorem, words in an algebra are of the form $v_1^{k_1} v_2^{k_2} \dots v_l^{k_l}$, where $l \leq H$ (H is the height of the algebra) and v_i belong to a finite set of words. Hence, each word in A is defined by its type – the oredered set $(v_1, \dots, v_l)$, and by the power vector $\vec{k} = (k_1, \dot{;} k_l)$. The set of different types is finite. The representability criterion is a set of conditions on types and power vectors. The type $(u_1, \dots, u_k)$ is called a subtype of the type $(v_1, \dots, v_l)$, if the sequence $(v_1, \dots, v_l)$ is of the form $(v_1, \dots, v_m, u_1, \dots, u_k, v_{k+m+1}, \dots, v_l)$, i.e., words of the type $(u_1, \dots, u_k)$ are subwords of words of the type $(v_1, \dots, v_l)$ (with the indication of the power decomposition). The words v_i will be considered noncyclic (i.e., they are not powers). Also we suppose that any number of words, which occur in the type $v_\alpha^{k_\alpha} \dots v_\beta^{k_\beta}$) in succession, cannot be represented as a product of a smaller number of powers. (This condition is not only on v_i, but on k_i also.)

Let us fix a sufficiently big number N. A component $v_i^{k_i}$ will be called *essential*, if $k_i > N$. As the essential height is equal to Gelfand-Kirillov dimension (Theorem 2.110), then the number of essential components is equal to Gelfand-Kirillov dimension also (if N is sufficiently big). Let us mark essential components in all products. Then we can represent the word $v_1^{k_1} \dots v_l^{k_l}$ as

$$D_0 v_{i_1}^{k_{i_1}} D_1 \dots v_{i_s}^{k_{i_s}} D_s,$$

where $k_{i_1}, \dots, k_{i_s} \geq N$ and D_α are products of nonessential components. If N is sufficiently big, then s is not greater, than the essential height. The representation in the above form can be made unique, if v_i are correct words (in the above defined sense). Let us note that powers cannot have big overlappings. If a word can be represented in the form $D_0 v_{i_1}^{k_{i_1}} \dots D_s$ in several ways, then we shall choose those, which corresponds to the minimal in length D_0, then the maximal in length $v_{i_1}^{k_{i_1}}$, then the minimal D_1 and so on. The obtained unique representation $(D_0, v_{i_1}, D_1, v_{i_2}, \dots, D_s)$ of a word is called the essential type (we omit the power vector). Let us sum our observations and formulate some useful remarks.

Proposition 6.24 *If N is sufficiently big, then to each word uniquely corresponds its essential type. There are only finite number of essential types. A subtype of the essential type $(D_0, v_1, \dots, v_s, D_s)$ can have one of the following forms: $(v_i, D_i, \dots, v_j)$, $(v_i, \dots, D_j'')$, $(D_{i-1}', v_i, \dots, v_j)$, $(D_{i-1}', \dots, v_j, D_j'')$, where D_{i-1}' is an end of $v_{i-1}^{k_{i-1}} D_i$ and D_j'' is a beginning of $D_j v_{j+1}^{k_{j+1}}$. Each word in a subtype is a subword of some word in the type. To each essential type corresponds a graph without parallel edges, such that all words in this type can be positioned in the graph. Moreover, v_i correspond to cycles and D_j correspond to paths, which connect them.* $\square$

Now we can formulate the representability criterion.

Theorem 6.25 *A monomial algebra A is representable, only when A has a bounded essential height and power vectors satisfy the following conditions.*

1. For each essential type a finite system of exponential Diophantine equations on power values of essential components is defined. Moreover

$$D_0 v_{i_1}^{k_1} D_1 \dots v_{i_s}^{k_s} D_s = 0$$

if and only if all these equations hold

$$P_{\alpha,t}(k_1, \dots, k_s) = 0$$

(by t the essential type $(D_0, v_{i_1}, \dots, v_{i_s}, D_s)$ is denoted).

2. If P_α is the system of equations for a subtype, then to it corresponds the system of equations for the type.

Proof. Let us prove the necessity, at first (and also let us clarify the last condition of the theorem). The last condition means that, if a word has a zero subword, then it is a zero word itself. The necessity of the essential height theorem was mentioned above.

To prove, why the validness of the system of exponential Diophantine equations is a consequence of the word equality to zero, we shall need the following proposition.

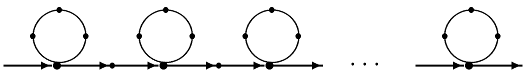
Proposition 6.26 *Let A be a square matrix. Then the (ij)-th component of its power A^n is of the form $\sum_{i=1}^{k} \lambda_i^n P_i(n)$, where λ_i is an element from a finite algebraic extension K of the ground field and $P_i \in K[x]$.*

Proof. This statement is a direct consequence of the theorem about Jordan cell. ☐

Hence, the equality to zero of a word of the given type means the equality to zero of components of the corresponding operator, i.e., the necessity can be proved by the removing of parentheses.

Let us come now to the sufficiency proving. Let the theorem condition holds. Our algebra is a quotient algebra of the algebra, which graph consists of several disjoint components, corresponding to different types. By the direct sum reasoning, it is enough to study the case of one type graph (and the system, which corresponds to this type). Proposition 6.8 reduces our problem to the case, when all arrows of the graph are marked by different letters. And Proposition 6.11 allows to restrict ourselfs to the case of one equation. So we can reformulate the problem.

Let A be an algebra of the following graph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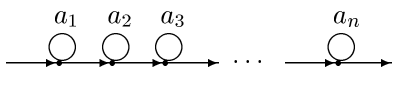
where all arrows are marked by different letters. Let $P(k_1, \ldots, k_s)$ be an arbitrary exponential Diophantine polynomial from $k_1, \ldots, k_s$. Let I be the ideal, which is generated by words positioned in a way, such that the first cycle is passed k_1 times, the second cycle – k_2 times and so on, and $P(k_1, \ldots, k_s) = 0$. We have to prove that the quotient algebra A/I is representable.

At first we shall reduce the statement to the case of unit loops and unit arrows, which connect loops. Let a representation, which corresponds to the quotient algebra of the graph



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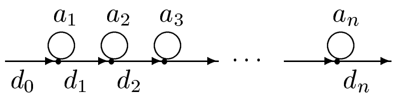
be constructed. And let V be the space of this representation. By $W_i \subset V$ will be denoted the image of the operator, which corresponds to the element a_i. Let us consider the direct sum of the space V and spaces $W_i^1, \dots, W_i^l$, isomorphic to W_i, and let us define operators $a_i^0, a_i^1, \dots, a_i^l$: a_i^j is an isomorphism from W_i^j to W_i^{j+1}, if $1 \le j < l$, on other components this operator acts by zero. The operator a_i^0 is the composition of a_i and an isomorphism from W_i to W_i^1. At last, the operator a_i^l is an isomorphism from W_i^l to W_i. Then the product $a_i^0 a_i^1 \dots a_i^l$ acts on V in the same way, as the operator a_i, i.e., the passage of a cycle corresponds to the action of a_i.

With the help of an analogous procedure we can obtain the situation, when cycles are connected by unit paths.

So, it remains to prove the following lemma.

Lemma 6.27 (on the representation) *Let us consider a graph Γ of the following form*



.

Let $P(k_1, \dots, k_n)$ be an arbitrary exponential Diophantine polynomial from variables $k_1, \dots, k_n$. If I is the ideal, generated by elements $d_0 a_1^{k_1} d_1 a_2^{k_2} \dots d_{n-1} a_n^{k_n} d_n$, such that $P(k_1, \dots, k_n) = 0$, then the algebra A_Γ / I is representable.

In the proof the technique of generalized graphs will be used.

6.4.1 Generalized graphs

Generalized graphs will be used for constructing examples of representable (but not automata!) monomial algebras with required properties.

Definition 6.28 1. By a generalized graph we understand an oriented graph with edges, marked by letters a_i and numbers λ_i.

2. The algebra of a generalized graph (and its representation) is constructed in the following way:

(a) to verteces of the graph correspond base vectors of the representation space;

(b) to each letter a_i corresponds its transcendental constant $\bar{a}_i$, which is the same for all positions of a_i in the graph;

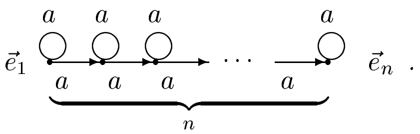
(c) to the arrow, which connects i-th vertex with j-th vertex and marked by the letter a_k and the number λ_s, corresponds the operator $\bar{a}_k \lambda_s E_{ij}$ (where E_{ij} is the matrix unit, i.e., the operator, which maps the i-th base vector e_i into j-th);

(d) to each generator a_i corresponds the sum of operators, where the summation is taken over all arrows, marked by a_i (if there is no outgoing arrow, marked by a_i, from some vertex, then the action of the operator on the corresponding base vector is zero, hence, if $\lambda = 0$, and we can delete the arrow);

(e) if an arrow is not marked by any number λ, then we assume that $\lambda = 1$.

Let us consider a Jordan cell A. How can we represent operators A and A^k by generalized graphs?

Proposition 6.29 *Let us consider the following generalized graph*



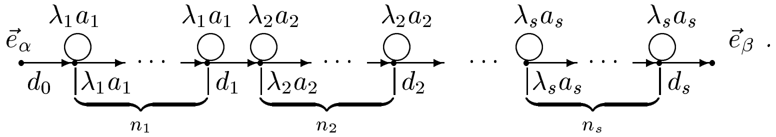
Let us denote by A the operator, which corresponds to the generator a. Then to A^k corresponds the graph, such that its arrows, which connect verteces $\vec{e}_i$ and $\vec{e}_j$, $i < j$, are marked by numbers $\binom{k}{j-i}\overline{A^k}$, where $\overline{A^k} = \bar{a}^k$ is the corresponding transcendental constant (we assume that $\binom{k}{0} = \binom{k}{k} = 1$, $\binom{k}{n} = 0$, if $n > k$).

Proof. This statement is a translation to the graph language of the well known fact about a power of a Jordan cell (which is easy to prove by the induction). □

We shall be interested in the arrow, which connects the extreme verteces $\vec{e}_1$ and $\vec{e}_n$. It is marked by $\binom{k}{n-1}\bar{a}^k$. If all arrows will be marked additionally by λ, then this arrow will be marked by $\lambda^k\binom{k}{n-1}\bar{a}^k$. We need transcendental constants to ensure the uniformity with respect to each variable.

The following proposition is a direct consequence of the previous.

Proposition 6.30 *Let us consider the algebra, which is defined by the generalized graph*



Let operators D_i correspond to generators d_i and operators A_i – to generators a_i, $i = 0, 1, \ldots$. Then

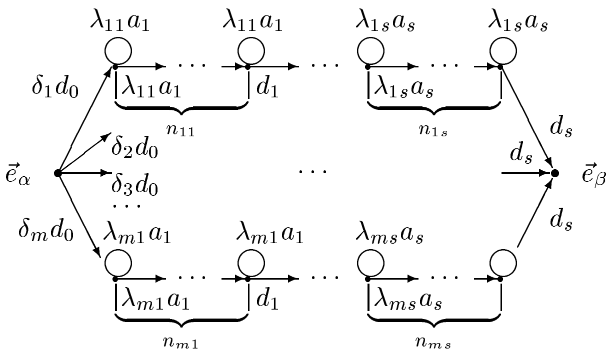
$$D_0 A_1^{k_1} D_1 A_2^{k_2} \ldots D_{s-1} A_s^{k_s} D_s = E_{\alpha\beta} \cdot \bar{d}_0 \prod_{i=1}^{s} \lambda_i^{k_i} \binom{k_i}{n_i - 1} \bar{a}_i^{k_i} \bar{d}_i \quad ,$$

where $E_{\alpha\beta}$ is a matrix unit. □

Proposition 6.31 *Polynomials of the form* $\binom{k_1}{n_1-1}\binom{k_2}{n_2-1}\dots\binom{k_s}{n_s-1}$, *for all systems* $n_1,\dots,n_s$, *constitute a base in the space of all polynomials* $K[k_1,\dots,k_s]$. *Polynomials with exponents* $\lambda_1^{k_1}\dots\lambda_s^{k_s}\binom{k_1}{n_1-1}\dots\binom{k_s}{n_s-1}$ *constitute a base in the space of all polynomials with exponents.* □

Remark. Polynomials $\binom{k_1}{n_1-1}\dots\binom{k_s}{n_s-1}$ constitute a $\mathbb{Z}$-base of polynomials from variables $k_1,\dots,k_s$, which have integral values in integral points.

Proposition 6.32 *a) Let us consider the following generalized graph*



Let operators A_i *correspond to generators* a_i *and operators* D_i *correspond to generators* d_i. *Then*

$$D_0A_1^{k_1}D_1A_2^{k_2}\dots A_s^{k_s}D_s = E_{\alpha\beta}\cdot\left(\bar{d}_0\prod_{j=1}^{s}\bar{a}_j^{k_j}\bar{d}_j\right)\cdot\sum_{i=1}^{m}\delta_i\prod_{j=1}^{s}\lambda_{ij}^{k_j}\binom{k_j}{n_{ij}-1}.$$

Therefore, $D_0A_1^{k_1}D_1\dots A_s^{k_s}D_s = 0$, *if and only if* $P(k_1,\dots,k_s) = \sum_{i=1}^{m}\delta_i\prod_{j=1}^{s}\lambda_{ij}^{k_j}\binom{k_j}{n_{ij}-1} = 0$.

b) To each word, which has a position in the graph (except, maybe a finite number of words, which connect the verteces $\vec{e}_\alpha$ *and* $\vec{e}_\beta$*), corresponds a nonzero operator. Nonzero operators, which correspond to different words, are linearly independent.* □

Proposition 6.32 allows to construct any linear combination of products of binomial coefficients, and Proposition 6.31 demonstrates that it is possible to construct any exponential Diophantine polynomial in this way. All this proves the representability lemma, hence the representability criterion is also proved.

□

6.4.2 Applications of the representability criterion

We shall use the representability criterion for proving that each representable algebra with Gelfand-Kirillov dimension 1 is automata. Also we shall use this criterion for constructing representable monomial algebras with patological properties.

Theorem 6.33 *A representable monomial algebra over a field of zero characteristic with Gelfand-Kirillov dimension* 1 *is automata and its Gilbert series is rational.*

Proof. This theorem is a consequence of the theorem about the coincidence of the essential height and Gelfand-Kirillov dimension, of the representability criterion and of the following theorem.

Theorem 6.34 *The set of integral zeroes of an exponential-polynomial Diophantine equation over a field of zero characteristic is a union of a finite number of points and a finite number of arithmetic series.*

Proof. We shall prove this theorem by Scolem method. Let us present at first some auxiliary facts from algebra and number theory.

1. From a set $\{\lambda_i\}$ of elements of a field (i.e., parameters, which are related to the equation under study) we can select a maximal system of algebraically independent elements $\{\lambda_1,\dots,\lambda_k\}$, which is called the transcendental base. Each λ_i is a root of an algebraic equation of the form $P_{i0}\lambda_i^n+\dots+P_{in}=0$, where all P_{ij} are values of polynomials with integral coefficients from $\lambda_1, dots, \lambda_k$. Each map of a transcendental base into a system of algebraically independent elements of a field H can be extended to a homomorphism of the field $\mathbb{Q}(\{\lambda_i\})$ into some algebraic extension of H.

2. Let us take a sufficiently big prime p, such that for some set of remainders $\{\bar\lambda_1,\dots,\bar\lambda_k\}$ all polynomials $P_{in_i}(\bar\lambda_1,\dots,\bar\lambda_k)$, $P_{i0}(\bar\lambda_1,\dots,\bar\lambda_k)\neq 0$. Let us take algebraically independent, over $\mathbb{Z}$, p-adic numbers $\{\lambda_1',\dots,\lambda_k'\}$, which remainders modulo p are equal to $\{\bar\lambda_1,\dots,\bar\lambda_k\}$ respectively. Then the map $\lambda_i\to\lambda_i'$ can be extended to an embedding of $\mathbb{Z}[\{\lambda_i\}]$ into some algebraic extension $\mathbb{H}$ of the ring of intergal p-adic numbers $\mathbb{Z}_p$. Moreover, the images of $P_{js}(\lambda_i)$ don't belong to the principal ideal $\mathbb{H}_p$ of the ring $\mathbb{H}$. Let m be the degree of $\mathbb{H}$ over $\mathbb{Z}_p$.

3. If λ isn't contained in the ideal $\mathbb{H}_p$, then $\lambda^{p^m-1}-1\in\mathbb{H}_p$. In this case the function t^x, where $t=\lambda^{p^m-1}$, is analytical by x.

4. The set of integral p-adic numbers is compact.

5. An analytical function, which has infinite number of zeroes in $\mathbb{H}$ is identical zero.

Let us begin the proof. Let $r=p^m-1$. Using steps 1-2, let us pass to the ring $\mathbb{H}$. We can decompose the set of powers $\mathbb{N}$ into r arithmetical series with a step r. Each such progression corresponds to the equation with base of powers λ_i^r (because the power changes with the step r), the left hand side of which is analytical. Therefore, each such series either contains only finite number of zeroes, or is identically zero.

Now we can give an example of a monomial semigroup, which is representable over a field of characteristic p and non-representable over a field of zero characteristic. Let us consider the equation for $x \in \mathbb{N} : (1+t)^x - t^x - 1 = 0$, where t is a transcendental element. Bases of powers belong to a field of characteristic p. Its solutions are $x = p^k$. Therefore, the semigroup with relations $AC = DA = DC = CD = 0$, $CA^nD =$, for $n = p^k$, is representable, by the representability criterion, over a field of characteristic p, and is non-representable over a field of zero characteristic, by the previous theorem.

Let us write matrix elements of its representation:

$$\begin{array}{ll} A: & (1+\lambda)aE_{11} + \lambda aE_{22} + aE_{33}, \\ C: & c(E_{01} - E_{02} - E_{03}), \\ D: & d(E_{14} + E_{24} + E_{34})\ . \end{array}$$

The Gilbert series of the corresponding monomial algebra is transcendental. It can be proved that this algebra is non-representable over each field of characteristic $\neq p$.

Let us give an example of a monomial algebra, which is representable over a field of zero characteristic and has Gelfand-Kirillov dimension 2. It is known that all solution of the Pelle equation $x^2 - dy^2 = 1$, $d \neq k^2$, are of the form $(x_n, y_n) : x_n + \sqrt{d}y_n = (x_0 + \sqrt{d}y_0)^n$, where (x_0, y_0) is the minimal solution. The set of all solutions has a logarithmic density and the Gilbert series of the, defined below algebra, is transcendental:

$$\begin{array}{l} CA^xB^yD = 0, \quad \text{if } x^2 - dy^2 = 0, \\ C^2 = D^2 = AC = BC = DC = DB = DA = 0, \\ BA = AD = CB = CD = 0\ . \end{array}$$

(This is the algebra of subwords of words of the form CA^xB^yD, where $x^2 - dy^2 \neq 0$.)

Hence, the problem about the representability of a monomial algebra can be reduced to the following problem: is the given set a set of zeroes of some exponential Diophantine polynomial from several variables? It is known that this problem is algorithmically unsolvable. The problem about the existence of a positive integral root of this polynomial is algorithmically unsolvable also. Therefore, the following statement holds.

Theorem 6.35 *The problem about the existence of isomorphism of two monomial algebras and the problem about the equality of their Gilbert series are algorithmically unsolvable.*

We can restrict ourselves to the set of 1000×1000 matrices over $\mathbb{C}[x_1, \ldots, x_9]$.

Remarks. The property of zeroes of an analytical nonzero function not to have limit points is the specifics of the one variable case. It is known that, the solving of a system of Diophantine equations from two variables, is an algorithmically solvable problem. If it is possible to generalize this result to the case of exponential Diophantine equations, then we shall get the algorithmical solvability of problems about representable monomial algebras with Gelfand-Kirillov dimension 2.

7 Varieties of monomial algebras

A variety of monomial algebras is a variety, which is generated by some set of monomial algebras.

We shall prove that such variety is generated by one finitely defined algebra with the graph without parallel edges and such that each arrow, which have a common vertex with a cycle, is marked by the unique letter, i.e., all other edges are marked by different letters. We shall get also a description of unitary closed varieties of monomial algebras: this are varieties, which are generated by direct sums of semidirect products of matrix algebras of arbitrary dimensions.

Varieties of monomial algebras constitute a "bridge" between structural properties of identities and word conbinatorial analysis. A general scheme of reasoning may be described in the following way: an identity f doesn't hold in a variety $\Rightarrow$ f doesn't hold in a monomial algebra $A \Rightarrow$ a word in an arbitrary algebra, which is graphically identical to a word in A, with the help of f can be transformed into a linear combination of other words. The last step allows to perform a reduction.

We shall apply this approach to matrix algebras and to algebras of upper triangular matrices. Also we shall prove lemmas, to which we refered in Section 2.2. In what follows, when we shall speak about words in $\mathrm{Wd}\langle A\rangle$, it means that we speak about words, graphically identical to words in A.

Proposition 7.1 *a) Let f be an identity, which doesn't hold in a monomial algebra A, and let B be an algebra with the same set of generators. Then there exists a word u in $\mathrm{Wd}\langle A\rangle$ in B, which is linearly representable modulo $T(f)$ by words, nonequal to u.*

b) Let f be a polylinear identity of the form $\sum_{\sigma\in S}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(p)}$, $\alpha_\sigma\in F$. Then there exist words $v_1,\ldots,v_p\in\mathrm{Wd}\langle A\rangle$ and coefficients β_σ, such that

$$\begin{array}{l} V=v_1v_2\ldots v_p\in\mathrm{Wd}(A),\\ v_1v_2\ldots v_p\equiv\sum_{V_\sigma\neq V}\beta_\sigma V_\sigma \quad (\operatorname{mod} T(f)), \quad \textit{where}\\ V_\sigma=v_{\sigma(1)}v_{\sigma(2)}\ldots v_{\sigma(p)}.\end{array} \tag{8}$$

Proof. If we take the factor by words, which dont't belong to $\mathrm{Wd}\langle A\rangle$, then we shall reduce the proof to the case of A itself. But this case is obvious. □

Let us note that this construction often ensures the linear representability of u, by words, which don't belong to $\mathrm{Wd}\langle A\rangle$.

Proposition 7.2 *Let $\mathfrak{M}$ be a unitary closed variety, f be an arbitrary identity of degree p, v_i, v_i' be words, which don't contain variables x_i. Then the validness of the identity $f(x_1, \ldots, x_p)$ in $\mathfrak{M}$ is equivalent to the validness of the identity $f(v_1x_1v_1', \ldots, v_px_pv_p')$.*

Proof. It is enough to substitute the unit instead of variables, which occur in v_i and v_i'. $\square$

Corollary 7.3 *Let* $\mathrm{Var}(A)$ *be a unitary closed variety. Then the words v_i in (8) can be simultaneously arbitrarily long.*

We see that, "deletions and addings" (see Lemma 2.6) are a combinatorial analog of the unitary closedness. Let us apply these results to a matrix algebra and to an algebra of upper triangular matrices. The following proposition was used in 2.2.

Proposition 7.4 *Let u be a noncyclic word of length n and let f be a polylinear identity of comlexity $< n$ and degree p. Then there exists a subword t in the superword u^∞, representable in the form $t = t_0 \ldots t_{p+1}$, where lengths of t_i satisfy the inequalities $2n \le |t_i| \le 3n$, and t is also representable as a linear combination of words $t_\sigma = t_0t_{\sigma(1)} \ldots t_{\sigma(p)}t_{p+1}$, which are not subwords in u^∞.*

Proof. By Theorem 5.18, $\mathrm{Var}(A_u) = \mathbb{M}_n$. If we apply the previous corollary to A_u, then, by Proposition 7.1, if $t_\sigma \neq t$, then $t_\sigma \notin \mathrm{Wd}\langle A_u\rangle$. The satisfaction of the inequalities $2n \le |t_i| \le 3n$ can be achieved with the help of the deletions and addings lemma. $\square$

Remark. The reasoning in Proposition 7.1 is not constructible. The constructibility may be achieved with the help of deletions and addings lemma.

Definition 7.5 Let us consider a nonmatrix variety $\mathfrak{M}$, which is generated by a f.g. algebra. By the *Latyshev complexity* $Lat(\mathfrak{M})$ of the variety $\mathfrak{M}$ is called the maximal n, such that the algebra of upper triangular matrices $\mathbb{T}_n$ belongs to $\mathfrak{M}$. The Latyshev complexity $\mathrm{Lat}(A)$ of a f.g. algebra A is the complexity of the variety $\mathrm{Var}(A)$, generated by A (in the case, when $\mathrm{Var}(A)$ is nonmatrix).

Let an algebra A be of the form

$$A = F\langle x, c_1, \ldots, c_n\rangle / \left(\sum_{i=1}^{n} \mathrm{id}(c_i)^2 + \mathrm{id}(c_1, \ldots, c_n)^n \right).$$

A is monomial and $\mathrm{Var}(A) = \mathrm{Var}(\mathbb{T}_n)$. Using Corollary 7.3, we have

Proposition 7.6 *If an identity f holds in an algebra A, and f doesn't hold in $\mathbb{T}_{n+1}$, then the weak algebraicity identity (and, by Proposition 2.95, the strong algebraicity identity also) of order $\le n$ holds in A.*

This proposition allows to pump over n powers of the same word into a smaller number of powers (see 2.2.3). The following proposition is useful in the estimation of Gelfand-Kirillov dimension.

Proposition 7.7 *a)* $\mathrm{GKdim}(A) \leq S \cdot \mathrm{Lat}(A)$*, where S is the minimal number of elements in a s-base.*

b) Let A be a f.g. algebra, which generates a nonmatrix variety. Then the strong algebraicity identity of order $\mathrm{Lat}(A)$ *holds in A and the strong algebraicity identity of order* $\mathrm{Lat}(A)-1$ *doesn't hold in A.*

Proof. The item a) is a consequence of Proposition 7.6 and the pumping over procedure: each element in A is linearly representable by words of the form $d_0 v_{i_1}^{e_1} d_1 v_{i_2}^{e_2} \ldots d_{h-1} v_{i_h}^{e_h} d_h$, where all words d_i have a bounded degree, words v_{i_j} belong to the s-base and for each element v from the s-base there can be not more, than $\mathrm{Lat}(A)$ words v_{i_j}, equal to v. Hence, $h \leq S \cdot \mathrm{Lat}(A)$.

The item b) is a consequence of Proposition 7.6 and those fact that the strong algebraicity identity of order $\mathrm{Lat}(A) - 1$ doesn't hold in $\mathbb{T}_n$. □

So, the Latyshev complexity and the ordinary complexity of varieties can be defined in the terms of monomial algebras. It is interesting to note, that more general "complexities" appeared in works, dedicated to the Specht problem. These complexities correspond to semidirect products of matrix algebras, i.e., to varieties, generated by unitary closed monomial algebras.

7.1 The reduction to the finitely generated case

Proposition 7.8 *Let a polylinear identity of degree m holds in a monomial algebra A. Then*

a) each word, which contains m different generators, is zero;

b) $\mathrm{Var}(A)$ *is generated by $(m-1)$-generated algebras and has base rank $\leq m-1$.*

Corollary 7.9 *The variety, which is defined by the identity $x^n = 0$, and also the variety, generated by Grassmann algebra, cannot be generated by monomial algebras.* □

Passing to a direct sum of $(m-1)$-generated algebras and using the diagonal embedding (see Proposition 6.6), we have the following theorem.

Theorem 7.10 *If an identity of degree m holds in a variety $\mathfrak{M}$ of monomial algebras, then there exists an $(m-1)$-generated monomial algebra A, such that* $\mathfrak{M} = \mathrm{Var}(A)$. □

7.2 The reduction to the automata case

Our next aim is to construct an automata algebra with the same set of identities, as an arbitrary f.g. monomial PI-algebra.

7.2.1 Verbal subalgebras

A subalgebra of a semigroup algebra, which is generated by words, is called a verbal subalgebra. A verbal subalgebra itself is a semigroup algebra.

Proposition 7.11 *A verbal subalgebra with generators $v_1, \ldots, v_k$ of a relatively free algebra is relatively free itself, only when the set of words $\{v_i\}$ is a code (i.e., if $v_{i_1} \ldots v_{i_k} = v'_{i_1} \ldots v'_{i_s}$, then $k = s$ and $v_{i_j} = v'_{i_j}$, for all j).*

A verbal subalgebra of a monomial algebra is monomial itself, only when its set of generators has the following property: if $W = v_{i_1} \ldots v_{i_k} = v'_{i_1} \ldots v'_{i_s}$, then either $W = 0$, or $k = s$ and $v_{i_j} = v'_{i_j}$, for all j. $\square$

In other words, a verbal subalgebra is monomial, if each nonzero word in it has a unique representation, as a product of generators. Let us note that the monomiality depends on the set of generators. The following proposition is a direct consequence of the previous one.

Proposition 7.12 *a) If the first letters in all v_i are differernt, then the verbal subalgebra with generators v_i is monomial. (An analogous statement for last letters also holds.)*

b) If $h_i = a_i \cdot \sum_j t_{ij}$ and $a_i \neq a_j$, if $i \neq j$, then the subalgebra with generators h_i is monomial. $\square$

Remark. Algebras, generated by monomials in a ring of commutative polynomials, studied by I.D.Gubeladze (see [19]). He proved that projective modules over such algebras are free.

7.2.2 Combinatorial properties of a set of words of bounded height

We studied periodic, quasiperiodic, (weakly) pseudoperiodic words. Now we need to study piecewise periodic words. This necessity is caused by the following reason. At first, by Shirshov height theorem, such words constitute a normal base in a PI-algebra. Secondly, the notion of piecewise periodicity is a natural generalization of the periodicity notion. Thirdly, to types of such words correspond connected graphs without parallel edges, and the type language corresponds to the graph language. The study of the piecewise periodicity leads to new notions and statements, which allow to give a description of representable monomial algebras. All this constitute a technique, necessary to study varieties of monomial algebras.

The definitions of a type, a subtype, an essential type and an essential height see in Section 6.4.

Definition 7.13 A set of words $\mathcal{R}$ is called *essentially connected with a given type $\mathcal{D}$*, if

a) each word from $\mathcal{R}$ has the essential type $\mathcal{D}$:

$$\forall w \in \mathcal{R} \quad w = d_0 v_1^{k_1} d_1 \dots v_s^{k_s} d_s \ ;$$

b) if $s \geq 1$, then $\forall N \exists w \in \mathcal{R}$: $\forall i \quad k_i > N$. If $s = 0$, then $\mathcal{R} = \{\mathcal{D}_\prime\}$.

Let us note that $\mathcal{R}$ doesn't necessary coincide with the set of all words in the given type. The fact that $\mathcal{R}$ is essentially connected with $\mathcal{D}$ means that $\mathcal{R}$ cannot be represented as a finite union of sets of words of smaller essential height.

Proposition 7.14 *Let a subset $\mathcal{R}$ in the set of words of the type $\mathcal{D}$ is not essentially connected with $\mathcal{D}$, then $\mathcal{R}$ can be represented as a finite union $\cup\mathcal{R}_\rangle$, where each $\mathcal{R}_\rangle$ has an essential type with a smaller essential height. (So,* $\mathrm{SH}(\mathcal{R}) < \int$*.)*

Proof. Let us suppose the contrary. Then some N satisfies the following condition: for each $w \in \mathcal{R}$, there exists i, such that $k_i < N$. Let $D_{ij} = D_i v_{i+1}^j D_{i+1}$, where $i = 0, \dots, s-1, j = 0, \dots, N-1$. Let us consider the set of types $\{(D_0, \dots, v_i, D_{ij}, v_{i+1}, \dots, D_s)\}$. This set is finite and each $w \in \mathcal{R}$ belong to one of these types. $\square$

The description by types means the isolation of powers $v_i^{k_i}$ and "spans" D_i. However, it is useful to isolate not only powers, but complete quasiperiodic segments.

Definition 7.15 By *a shape* will be called a sequence

$$\mathcal{D} = (d_0, v_1, v_1', d_1, v_2, v_2', \dots, v_s, v_s', d_s),$$

where v_i and v_i' are cyclically conjugate. A word W is of the shape $\mathcal{D}$, if $W = d_0 t_1 d_1 \dots t_s d_s$, where $t_i \subset v_i^\infty$ and t_i is a beginning of $v_i^{\infty/2}$ and an end of $(v_i')^{\infty/2}$. (In the first case the right superwords are considered, in the secod – the left.)

A shape is called convenient, if an end of d_0 doesn't coincide with a beginning of v_0, a beginning of v_i' doesn't coincide with a beginning of d_i and an end of v_i doesn't coincide with an end of d_{i-1}. (Let us note that some d_i can be empty words.) A shape will be called completely convienient, if in the case, when $d_i = \Lambda$, the first letter of v_i is different from the first letter of v_i'. The meaning of the conveniency notion is in the maximality of quasiperiodic segments, related to v_i.

The notion of the essential connectivity is defined in the same way, as for types. The definition of the correspondence between types and shapes is obvious. The following proposition can be proved in the same way, as previous.

Proposition 7.16 *a) To each type the completely convenient shape is uniquely corresponded.*

b) A set of words of a bounded height over a finite set of words can be represented as a finite union of sets, essentially connected with a completely convenient shape. □

Corollary 7.17 *Each set $\mathcal{R}$ of words of a bounded essential height over a finite set Y can be represented as a finite union $\mathcal{R} = \cup\mathcal{R}_{\rangle}$, where each set $\mathcal{R}_{\rangle}$ is essentially connected with some type.* □

We are mainly interested in places, where a quasiperidic segment ends and another quasiperiodic segment begins.

Definitions and constructions. A shape is called open to the left, if $d_0 = \Lambda$, it is called open to the right, if $d_s = \Lambda$, and it is called open, if it is open both to the right and to the left. The openness of a type is defined by the openness of the corresponding shape. The definition of the closedness (the right, the left, the bilateral) is obvious. A left place in a shape $\mathcal{D}$ is the position of the first letter of d_i (and this letter itself), a right place – is the position of the last letter in d_i (and this letter itself), a bilateral place is the place, which is simulteneously the right and the left, or a letter with its position, which can occur in both successive quasiperiodic segments, which correspond to different t_i in types. The identity of places is defined by the coincidence of the corresponding letters and periods and with the correspondence of the class of the cyclic conjugacy of the period, which is next to the marked position. A substitution, corresponding to a left place is of the form $bv^k \to b$, where v is the quasiperiod, which begins after b. Substitutions, which correspond to a right place, are defined analogously. A substitution of a bilateral place is of the form $v^k bu^l \to b$. We shall lengthen words by substitutions in possible positions. Only those substitutions will be of interest to us, which preserve the shape,

Proposition 7.18 *A substitution, corresponding to some place, with sufficiently big powers of pasting in periods, preserves the shape, only when the position of this substitution is identical to the place of this substitution.* □

An analogous statement is valid for a system of simultaneous substitutions.

7.2.3 On pastings in and substitutions

When we use substitutions, the question arises about the preservance of the shape under the operation of the pasting in a segement.

Proposition 7.19 *a) Let u be a noncyclic word, v be cyclically conjugate to u^k and a be a letter in a word s. Then the substitution $av \to a$ transforms s into a subword in u^∞, only when $s \subseteq u^\infty$. In this case s can be positioned in u^∞*

in a way, such that after the letter a the period v begins. If $|s| \geq |u|$, then all possible positions of the substitution $av \to a$, which preserve the property "to be a subword in u^∞", are situated in s with the period $|u|$.

b) Let bu^k be not a subword in u^∞. If s is a beginning of $bu^{infty/2}$ and the substitution $bu \to b$ is performed in the first position, then this substitution transforms s into a subword in $bu^{\infty/2}$. An analogous statement holds for $u^{\infty/2}b$.

c) Let $W = v^{\infty/2}cu^{\infty/2}$ and $W \not\subset u^\infty$. Then $W \not\subset v^\infty$. Let $|v^k| > 2|u|, |u^k| > 2|v|$, then the substitution $vcu \to c$ transforms a word s into a subword in W, only when $s \subset W$ and s is of the form $s = v'cu'$, where v' is an end of $v^{\infty/2}$ and u' is a beginning of $u^{\infty/2}$. If $|v'| \geq |v|$ or $|u'| \geq |u|$, then the representation of s in the above form is unique. □

Definition 7.20 Substitutions from a), b) and c) of this proposition will be called *inner substitutions, substitutions of the right (left) end, substitutions of the bilateral end*, respectively.

7.2.4 Formal power series

We consider $\mathbb{N}$-graded algebras, which are direct sums of their homogeneous components. As usual, $A_nA_m \subseteq A_{m+n}, A = \oplus_{k=0}^{\infty} A_k$. We can define formal power series in such algebras. A formal power series is an infinite sum $\sum v_i$ of homogeneous elements v_i, such that the number of elements of each degree of uniformity is finite. The set of all formal power series has a natural structure of algebra, which is denoted by $\widehat{A}$. The algebra A has a natural embedding into $\widehat{A}$.

Proposition 7.21 *a) $A/A^n = \widehat{A}/\widehat{A}^n, \forall n$.*

b) $\mathrm{Var}(\widehat{A}) = \mathrm{Var}(A)$.

c) The operator $\hat{}: A \to \widehat{A}$ is a functor from the category of graded algebras into the category of algebras. □

The following proposition is a consequence of Propositions 7.11, 7.21.

Proposition 7.22 *Let a_i be generators of a monomial algebra A, $t_i \in \widehat{A}$, $\lambda_i \in F$, where F is the ground field. Let $a_i' = \lambda_i a_i + a_i t_i$ and A' be the algebra, generated by a_i'. Then A' is monomial. If all $\lambda_i \neq 0$, then to each nonzero word in A corresponds a nonzero word in A' and $\mathrm{Var}(A') = \mathrm{Var}(A)$.* □

This correspondence between a word v in A and a word v' in A' will be denoted by $'$.

The idea of the construction of an automata algebra, which define the given variety is as follows. Words of A are divided into sets of words, essentially connected with types. For each type $\mathcal{D}$ we shall find t_i, such that in A' all words of this type will be nonzero. The set of all subwords of this words constitute a regular language, i.e., it is automata.

The procedure of checking, that a word $W' \in \mathrm{Wd}(A')$ is nonzero, is as follows: if we substitute formal power series and remove the parentheses, then we shall have a nonzero word v in A. We also need to check (especially in characteristic p) that we shall have v with a nonzero coefficient. For this we shall prove the uniqueness of v construction with respect to a given set of simultaneous substitutions.

Using properties of subwords of a periodic word (Proposition 2.3) and the induction on the length of a word, we have

Proposition 7.23 *Let $c_i = v_i^{k_i} b_i$, where $k_i \geq 2$, v_i doesn't begin with the letter b_i and the inequality $|v_i^{k_i}| \geq |v_j^2|$ holds, for all i, j. Let $W' = s_0 c_1 s_1 \ldots c_k s_k$ and, for all sigma $\in S_k$, let $W'_\sigma = s_0 c_{\sigma(1)} s_1 \ldots c_{\sigma(k)} s_k$. Then, if $W'\sigma = W'\tau$ (where $\sigma, \tau \in S_k$), then $c_{\sigma(i)} = c_{\tau(i)}$, for all i.* □

Remark. The proposition is wrong, if words c_i are generic (even, if they are not weakly pseudoperiodic): usually we can choose in two ways "constant segments" s_i in W'. But, as usual, substitutions are controlled, hence, we can think about a generalization.

We shall need a stronger result.

Proposition 7.24 *a) If a given word W' is produced from W by simultaneous substitutions $c_i \to b_i$, then the positions of these substitution can be computed from the known W'.*

b) Let each letter from the set of letters $\{b_1, \ldots, b_k\}$ occurs only once in the set of words $\{\bar{b}_1, \ldots, barb_r\}$ and, moreover, other letters don't occur in words of this set. By N_{c_i} will be denoted the number of substitutions, equal to $c_i \to b_i$, and let λ_{c_i} be algebraically independent (over $\mathbb{Q}$ or $\mathbb{Z}_p$) elements of the ground field F, which correspond to words c_i. Let $b'_j = \bar{b}_j + (\sum_{b_i = \bar{b}_j} \bar{b}_j$. Then, after simultaneous substitution $b'_j \to \bar{b}_j$ and removing the parentheses, terms with the coefficient $\prod_{c_i} \lambda_{c_i}^{N_{c_i}} = \prod_{i=1}^{k} \lambda_{c_i}$ are all those terms, which are obtained by the above defined substitutions. The right hand product is taken for all i, the left had – for different c_i (therefore, the multiplicities N_{c_i} appeared).

Proof. The item b) is a direct consequence of a). Let us prove a) by the induction on the length of W. If a left end of W' doesn't coincide with some c_i, then there was no substitution in this position and we can delete the left symbol from W and W'. Let $v_i^m b_i$ and $v_i^l b_i$ be the ends of W and W' respectively and powers m and l are as big, as possible. If $m > l - 2$, then there was no substitution in the last position, and we can make the descent. If $l \geq m + 2$, then there was no substitution inside v_i^m (see Proposition 7.23). Therefore, the substitution $v_i^{l-m} b_i \to b_i$ was performed in the last position, i.e., c_i is defined $c_i = v_i^{l-m} b_i$. So, we can delete the last symbol from W and the end c_i from W' and make the descent. □

By summing the results of Propositions 7.24 and 7.22, we have

Proposition 7.25 *Let elements $\alpha_{ij} \in F$, $i = 1, \ldots, s$, $j = 0, 1, \ldots, \infty$, be algebraically independent and let* $\mathrm{Wd}(A) = \{w_j\}_{j=0}^{\infty}$ *be the set of all words in A. Let $a_i' = a_i + a_i \cdot \sum_{j=0}^{\infty} \alpha_{ij} w_j$. If* $\mathrm{Wd}(A)$ *contains a subset $\mathcal{R}$ of words, which is essentially connected with the essential type $\mathcal{D} = (\lceil_\imath, \sqsubseteq_\infty, \lceil_\infty, \ldots, \sqsubseteq_\updownarrow, \lceil_\updownarrow)$, then all words of the type $CalD$ in A' are zero.* □

Remark. Let $\mathrm{Wd}(A'')$ is the union of $\mathrm{Wd}(A)$ and the set of all subwords of words of the type $\mathcal{D}$. Then $\mathrm{Var}(A') \supseteq \mathrm{Var}(A'') \supseteq \mathrm{Var}(A)$, hence, $\mathrm{Var}(A'') = \mathrm{Var}(A)$.

Now we are ready to prove the following theorem.

Theorem 7.26 *Each variety $\mathfrak{M}$ of monomial algebras is generated by an automata algebra.*

Proof. There exists a finitely generated monomial algebra A, such that $\mathfrak{M} = \mathrm{Var}(A)$. By Corollary 7.17, $\mathrm{Wd}(A)$ is a finite union of sets $\mathcal{R}_\rangle$ and each $\mathcal{R}_\rangle$ is essentially connected with a type $\mathcal{D}_\rangle$. A can be embedded into the finite direct sum of quotient algebras A_i (the diagonal embedding), which correspond to sets of subwords of words from $\mathcal{R}_\rangle$. But a finite union of regular languages is regular. Therefore, the proof is reduced to the case when $\mathrm{Wd}(A)$ is the set of all subwords of words from one $\mathcal{R}_\rangle$.

By the previous remark, we can assume that $\mathrm{Wd}(A)$ consists of all subwords of words of the type $\mathcal{D}_\rangle$. But this set is a regular language, hence A is automata. □

7.3 The description of varieties of automata algebras in graph terms

Definition 7.27 A graph is called *remarkable*, if it is finite, connected, if it doesn't have parallel edges, ingoing and outgoing verteces of its cycles coincide and each arrow, which has a common vertex with a cycle, is marked by a letter, such that no other arrow is marked by this letter. A graph is called *piecewise remarkable*, if it is finite, each its connected component is remarkable and arrows from different components are marked by different letters. In such graph all cycles are irreducible.

Proposition 7.28 *Each piecewise remarkable graph defines a finitely defined algebra.* □

Our next aim is to prove the following theorem.

Theorem 7.29 *Each variety $\mathfrak{M}$ of monomial algebras is generated by an algebra of some piecewise remarkable graph.*

In other words, $\mathfrak{M}$ is a finite union of varieties, defined by algebras of remarkable graphs.

Corollary 7.30 *Each variety of monomial algebras is generated by a finitely defined algebra.* $\square$

For proving the theorem we shall need some auxiliary statements.

7.3.1 On graphs without parallel edges, which generate PI-algebras

The description of varieties, generated by automata algebras, can be reduced to the case of automata algebras, which are defined by graphs without parallel edges. (An arbitrary variety is a union of such varieties.) In the PI-case there are no linking cycles and each graph Γ without parallel edges is an ordered chain of cycles, with cross connections. There is a correspondence between graphs – "chains of cycles" and shapes of words. In the description of piecewise periodic words the language of such graphs is parallel to the language of shapes, therefore, it deserves the study. Let us note, that the statement about the decomposition of the set of piecewise periodic words into a finite number of shapes correspond to the statement about the representation of a language, as a finite disjoint union of graphs without parallel edges. The study of such graph can be reduced to the "local" study of passage of paths inside its fragment. A fragment is a cycle (or a pair of cycles) with fixed initial (ingoing) and final (outgoing) verteces.

Merged cycles. Two cycles are called *merged*, if some superword u^∞ can be positioned on both of them. In opposite case, these two cycles are called *separated*. Two cycles are called *adjacent*, if there exist a *cross connection*, connected them, i.e., the shortest path, which connect them, doesn't contain any verteces from other cycles. If the length of the cross connection is 1, then this cycles are called neighboring.

Proposition 7.31 *a) Let C_1 and C_2 be merged cycles and Γ be their union with the cross connection. Then C_1 and C_2 have common small period u and all words of length $|u|$, which have a position in Γ, are cyclically conjugate to u. The inverse statement also holds.*

b) Two words of the same length, which have positions with the same initial arrow, coincide. A determinate graph doesn't have any merged cycles. $\square$

As each language is defined by a determinate graph and, the deleting of arrows and the passage to a disjoint union of graphs without parallel edges, preserve the determination, then each graph can be reduced to a graph without parallel edges and merged cycles.

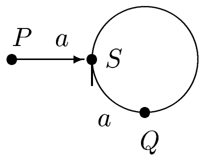
For further reduction we shall need to study operations over graphs, which are defined in following propositions.

Proposition 7.32 *Let a graph Γ' is produced from a graph Γ by the lengthening of a cycle C, i.e., by pasting in a power of its big period (the small period remains the same). Then* $\mathrm{Wd}(\Gamma) \supseteq \mathrm{Wd}(\Gamma')$. □

Remark. If we paste in the small period, then the proposition is wrong.

Proposition 7.33 *With the help of the pasting in operation, we can move off ingoing verteces from outgoing on an arbitrary big distance. If Γ doesn't have any adjacent merged cycled, then, by the moving off ingoing verteces from outgoing, we can liquidate merged cycles.* □

Proposition 7.34 *a) Let Γ be of the form*



(verteces P and Q don't belong to other cycles, Q is not an outgoing vertex, the arrow PS and QS are marked by the same letter). Let the graph Γ' is the result of pasting of P, Q and arrows, then $\mathrm{Wd}(\Gamma) = \mathrm{Wd}(\Gamma')$.

b) The dual statement, which can be obtained from a), by changing the direction of arrows, is also valid. (In the determinate case such pasting is impossible.)

c) If in the item a) we delete the condition "Q is not an outgoing vertex" (respectively, in b) – "Q is not an ingoing vertex"), then $\mathrm{Wd}(\Gamma) \subseteq \mathrm{Wd}(\Gamma')$.
□

Definition 7.35 A graph without parallel edges is called *not bad*, if it doesn't have any adjacent merged cycles and, if any two arrows, which go into the same ingoing vertex of some cycle, are marked by different letters (except, may be, the case, when one of these arrow connect two cycles). The same is true for outgoing arrows. A not bad graph is called *good*, when it is determinate.

Proposition 7.36 *a) With the help of the operations, defined in items a) and b) of the previous proposition, and also using the pasting in cycles, each graph Γ can be transformed into a not bad graph Γ'. If Γ is determinate, then Γ' will be good. If Γ is not determinate, then we shall need operations, inverse to operations from items a) and b), to obtain a good graph. Also we have that* $\mathrm{Wd}(\Gamma') \subseteq \mathrm{Wd}(\Gamma)$.

b) If, instead of pasting in cycles, will be used the operation from the item c) of the previous proposition, then Γ can be transformed into a not bad graph Γ''. If Γ is determinate, then Γ'' will be good. And $\mathrm{Wd}(\Gamma'') \supseteq \mathrm{Wd}(\Gamma)$.

c) These graphs $\Gamma, \Gamma', \Gamma''$ have the same small periods and graphs Γ', Γ'' have isomorphic fragments of cross sections, which don't have common verteces with cycles. □

We can represent cycles, as “drums”, and cross sections, as threads, which wind around “drums”. Those verteces, where ingoing and outgoing arrows are marked by letters, different from letters of the corresponding cycle, constitute “knots”, which don’t allow “drums” to move. The previous proposition simply allows to rotate drums up to knots.

Remark 1. We have: $\mathrm{Var}(A_{\Gamma''}) \supseteq \mathrm{Var}(A_\Gamma) \supseteq \mathrm{Var}(A_{\Gamma'})$. We shall prove that $\mathrm{Var}(A_{\Gamma''}) = \mathrm{Var}(A_{\Gamma'})$, hence, $\mathrm{Var}(A_{\Gamma''}) = \mathrm{Var}(A_\Gamma) = \mathrm{Var}(A_{\Gamma'})$.

Remark 2. The inversion of operations from a) and b) of Proposition 7.34 is not needed, if Γ is determinate. To reduce a graph to a good form, we use the determination, but it is possible to describe explicitly the transformation of a graph with merged cycles. Words, which are positioned along a chain of adjoint merged cycles with small period u and multiplicities n_i, are of the form $u^{\sum k_i n_i} v$. And this set of words corresponds to the graph with parallel edges and the cycle of multiplicity $gcd(n_i)$. This graph can be transformed into a disjoint union of graphs without parallel edges. The operations from Proposition 7.34 transform each component into graphs, such that only ingoing and outgoing arrows are different in them. We shall see that corresponding algebras define equal varieties.

So, it is enough to study only good graphs. Our aim is to prove that $\mathrm{Var}(A_{\Gamma''}) = \mathrm{Var}(A_{\Gamma'})$. As $\mathrm{Var}(A_{\Gamma''}) \supseteq \mathrm{Var}(A_{\Gamma'})$, we have to check that $\mathrm{Var}(A_{\Gamma''}) \subseteq \mathrm{Var}(A_{\Gamma'})$. To prove this, we shall obtain equal varieties, by not diminishing $\mathrm{Var}(A_{\Gamma''})$ and not increasing $\mathrm{Var}(A_{\Gamma'})$.

By the quasiperiodicity of solutions of the word equation $Ax = xB$ (see Proposition 2.7), we have

Proposition 7.37 *Let $\{v_i\}$ be a set of words with undounded quasiperiods and let $t = \sum \alpha_i v_i \in \widehat{A}$. Then the element t and generators of A generate a monomial algebra.* $\square$

Let b be not a beginning of u. If we take the set $u^n b$ instead of $\{v_i\}$, then we have the following proposition.

Proposition 7.38 *Let Γ be a graph without parallel edges and linking cycles, let C be its cycle and v be an outgoing from C arrow, marked by b, and b is different from those letter, which marks an arrow in C with the same beginning vertex, as the beginning vertex of v. Let Γ' be the result of the following sticking letters operation in Γ: v is marked by a letter, which is not occured anywhere else, and there are no other changes in Γ. Then $\mathrm{Var}(A_{\Gamma'}) = \mathrm{Var}(A_\Gamma)$. The analogous statement about an ingoing arrow also holds.* $\square$

Using this proposition, we can reduce the variety, generated by an algebra of a good graph, to the very good case.

Let us study now positions of words.

Proposition 7.39 *Let Γ be a good graph, such that ingoing and outgoing verteces of cycles are separated from each other by a sufficiently big number of small periods. Let u be a noncyclic word, v be a cross section between adjoint cycles C_1 and C_2 with small periods u_1 and u_2, respectively. Let the word u_1vu_2 has the following position: v is in cross section and each u_i on its cycle. Then the following statement hold, for k sufficiently big:*

a) for any position of the word $u_1^{k_1}vu_2^{k_2}$, the word v is in the cross section and u_i are small periods of the corresponding cycles;

b) if b is not an end of u, then bu^k can be positioned only in the following way: b is in an ingoing segment of a cycle and u completely in this cycle;

c) a word u^k can have not more, than two positions with a fixed end. Each position of u^k is either in a cycle C with the small period u, or it begins in an ingoing cross section or in previous cycle C'. Moreover, the ingoing vertex of C' cannot belong to the position of u^k (here we use the condition that the distance between ingoing and outgoing verteces of a graph is greater, than the small period);

d) if b is not a beginning of u, then, in a position of u^kb, b is in the outgoing, from the cycle, part of the cross section (and the position of u^k is the same, as in the item c)). □

Let us translate this proposition to the language of representations. Let us consider, at first, the case of a cyclic graph with n verteces. All its verteces (and, hence, base vectors of the canonical representation) are marked by $\mathbb{Z}_n$. The following statement (which is a consequence of Proposition 2.3) is about positions of words in the cycle.

Proposition 7.40 *a) Let the cycle C be irreducible and u be its period, $n = |u|$, $v \subset u^\infty$, $|v| \geq n$, $v = v_0^k s$, $|s| < n$, $|v_0| = n$. Then v has the unique position in the cycle and in the canonical representation the operator $t^k\bar{s}\cdot E_{i,i+|s|}$ corresponds to v. Here i is the beginning vertex of v position, $i+|s| = i+|v|$ is the last vertex, t is the product of constant, written on arrows along C, $\bar{s}$ is the product of constants, written on arrows along the position of s with beginning in i-th vertex.*

b) Let the cycle C is reducible, u be its small period, u^l be its big period. Let $v \subset u^\infty$, $|v| \geq |u|$, $v = v_0^{xl}v_0^q s$, $|s| < |u|$, $|v_0| = |u|$, $q < l$. Then all positions of s differs from each other by rotations by $|u|$. To v corresponds the following operator in the canonical representation

$$t^x \cdot \sum_k \bar{s}_k E_{\alpha\beta}, \quad \alpha = i + k|u|,\ \beta = i + k|u| + q|u| + |s|,$$

where t is the product of constants, written on arrows along C and $\bar{s}_k$ is the product of constants, written on arrows along the position of $v_0^q s$ with the number of the beginning vertex $\alpha = i + k|u|$. □

Proposition 7.41 *a) Let C be a cycle and u be its small period. Then each central extension of the canonical representation of A_C contains all operators of the form $\sum_k E_{i+k|u|,i+k|u|}$ and also operators of the form $\sum_k \bar{s}_{i+k|u|,j+k|u|} E_{i+k|u|,j+k|u|}$, where $\bar{s}_{ij}$ is the product of constants along the shortest path, which connects i-th and j-th verteces.*

b) Let Γ be a disjoint union of cycles. Then some central extension of algebra of operators of canonical representation, contains all operators of the defined above form for each cycle in Γ (the sum is taken for verteces of only one cycle).

c) Let Y be a subset in the set of generators of an algebra A, and let arrows of the graph Γ, marked by letters from Y, constitute a disjoint union of cycles C_i and lines. Then the statement of the item b) holds for Γ and cycles C_i.

Proof. The item c) is a direct consequence of the item b). The item a) is a consequence of the previous proposition. The item b) also is a consequence of the previous proposition and the algebraic independence of constants of the canonical representation, which correspond to different cycles. □

7.3.2 Very good graphs

A graph Γ is called *very good*, if each arrow, ingoing into a cycle or outgoing from it, is marked by a letter, which doesn't occur in any other position. When we work with such graphs, then there are no difficulties with positions of words in different cycles, The following proposition is useful in the reduction to the case of the coincidence of ingoing and outgoing verteces of cycles.

Proposition 7.42 *a) Let Γ be a very good graph, C be its cycle with B, as the ingoing vertex, E, as the outgoing vertex, and an arbitrary vertex F. Let the ingoing into C arrow be marked by t_1 and the outgoing arrow – by t_2. Let the word s corresponds to the path from B to F, $t_1' = t_1 s$. Let A' be a subsalgebra in A_Γ, generated by all generators of A_Γ, different from t_1, and the element t_1'. Then A' is monomial, automata and is defined by a very good graph Γ', which differs from Γ only in the position of one arrow, marked by t_1'. The end of this arrow is in the vertex F.*

The analogous statement holds for the outgoing vertex and the substitution $rt_2 = t_2' \to t_2$.

b) All algebras of very good graphs, which differ from each other only in positions of ingoing and outgoing verteces of cycles, can be embedded into each other and, therefore, generate identical varieties. □

The following proposition is a consequence of Proposition 7.38.

Proposition 7.43 *a) Let Γ' can be produced from a graph Γ, by sticking the letters, such that the equlity of letters inside cycles and in the set of all cross sections is preserved and letters in different cycles or in cycles and cross sections become different. Then the algebras A_Γ and $A_{\Gamma'}$ have isomorphic central extensions. Therefore,* $\mathrm{Var}(A_\Gamma) = \mathrm{Var}(A_{\Gamma'})$.

b) Let Γ'' can be produced from Γ' by sticking the letters inside small periods (i.e., we don't change letters in cross sections, and inside cycles only those letters can coincide, which positions differ by a small period multiple). Then the algebras $A_{\Gamma'}$ and $A_{\Gamma''}$ have isomorphic central extensions and $\mathrm{Var}(A_{\Gamma'}) = \mathrm{Var}(A_{\Gamma''})$.

Let us now reduce the problem to the case, when all cycles are irreducible.

Proposition 7.44 *Let $\Gamma = C$ be a cycle, n be the length of its small period, nl be the length of the big period, let letters $x_1, \ldots, x_n$ be different and $x_1x_2\ldots x_n$ be the small period of Γ, let $x'_n = x_n(x_1x_2\ldots x_n)^{l-1}$. Then $x_1, \ldots, x_{n-1}, x'_n$ generate a monomial algebra, isomorphic to A_Γ. If we mark positions of generators of this algebra in Γ, then we shall obtain the graph Γ', which consists of l identical nonintersecting cycles, such that in each cycle arrows, marked by x_i, $i < n$, are the same, as in Γ, and arrows, marked by x'_n, correspond to passage from the end of each small period to its beginning.* $\square$

Proposition 7.45 *Let a graph Γ'' satisfies the condition of Proposition 7.43 and, moreover, ingoing and outgoing verteces in cycles coincide. Let Γ' be produced from Γ'' by replacing cycles by irreducible cycles with the same small periods. Then*

a) $\mathrm{Wd}(\Gamma') \supseteq \mathrm{Wd}(\Gamma'')$*;*

b) the algebra $A_{\Gamma'}$ can be embedded into $A_{\Gamma''}$;

c) $\mathrm{Var}(A_{\Gamma'}) = \mathrm{Var}(A_{\Gamma''})$.

Proof. The item c) is a direct consequence of a) and b). The item b) is obvious. Let us prove a). The set $\mathrm{Wd}(\Gamma')$ is the set of subwords of words of the form

$$c_0u_1^{k_1}c_1\ldots u_l^{k_l}c_l, \tag{9}$$

where c_i correspond to cross sections and u_i to small periods of cycles. (If the first cycle doesn't have any ingoing verteces, then $c_0 = \Lambda$, if the last cycle doesn't have any outgoing verteces, then $c_l = \Lambda$.) The set $\mathrm{Wd}(\Gamma'')$ is the set of subwords of words of the form

$$c_0u_1^{n_1k_1}c_1\ldots u_l^{n_lk_l}c_l, \tag{10}$$

where n_i are multiplicities of small periods in cycles of Γ''. But each word of the form (10) is of the form (9) also. Hence $\mathrm{Wd}(\Gamma') \supseteq \mathrm{Wd}(\Gamma'')$. $\square$

Let us sum the obtained above results.

Proposition 7.46 *Let Γ be a very good graph and let Γ' can be produced from Γ by the following operations.*

1. The translation of ingoing verteces of cycles to obtain the coincidence of ingoing and outgoing verteces.

2. The replacement of all cycles by irreducible cycles with the same small periods.

3. The sticking of letters: marks in cross sections are not changed; marks inside cycles are changed so, that each arrow inside each cycle is marked by a unique letter.

Then $\mathrm{Var}(A_{\Gamma'}) = \mathrm{Var}(A_\Gamma)$. □

7.3.3 The proof of the main theorem

Let us come to the proof of Theorem 7.29. It consists of several steps. At first, a variety $\mathfrak{M}$ of monomial algebras is generated by an automata algebra A. By the closedness of varieties with respect to the direct sum operation and by Proposition 6.6 about the diagonal embedding, the graph Γ_A can be assumed to be a connected graph without parallel edges. In Proposition 7.36 a good graph Γ' and not bad graph Γ'' were constructed, such that

1) $\mathrm{Wd}(\Gamma'') \supseteq \mathrm{Wd}(\Gamma_A) \supseteq \mathrm{Wd}(\Gamma')$;

2) big periods in cycles of Γ'' coincide with small;

3) there exists the correspondence between Γ' and Γ'': cycles correspond to cycles in the same order and with the same small periods; subgraphs, which don't have common verteces with cycles, are isomorphic (in the category of marked graphs). The position of a cross section between cycles is preserved (i.e., the corresponding cross section is positioned between the corresponding cycles) and lengths of cross sectiones are preserved also.

In other words, Γ'' and Γ' can differ only in big periods, in positions of ingoing and outgoing verteces and in marks on ingoing and outgoing arrows.

The theorem will be proved, if we shall transform Γ'' and Γ' into one remarkable graph Γ, by not diminishing the variety $\mathrm{Var}(\Gamma'')$, in the first case, and by not increasing $\mathrm{Var}(\Gamma')$, in the second.

Let us start with the simplest transformation. By the sticking of letters in Γ'', let us achieve the situation, when each arrow, which has a common vertex with a cycle, is marked by a unique letter. It remains to achieve the coincidence of ingoing and outgoing verteces in cycles. But this can be done with Proposition 7.42.

Let us turn now to Γ'. By Proposition 7.38, we can, by sticking of letters in arrows (and not chaging the variety), transform Γ' into a very good form. Proposition 7.46 allows to transform Γ into a remarkable form (by translating ingoings to outgoings, by replacing cycles by irreducible cycles, by sticking of letters inside cycles). It is easy to see that thus obtained graphs Γ'' and Γ' coincide. Theorem is proved. □

Remark. The coincidence of varieties, generated by algebras, can be proved in many ways. We can study canonical representations and prove isomorphisms of suitable central extensions of operator algebras. We can use the fact that an algebra and its Zariski closure generate the same varieties. Instead of transforming graphs, we can study positions of words, which "strongly participate" in minimal possible number of cycles (there only finite number of such positions), and then use central extensions or Zariski closure.

Let us use the proved above theorem for the classification of unitary closed varieties of monomial algebras.

Theorem 7.47 *Let $\mathfrak{M}$ be a variety of algebras, defined by a piecewise remarkable graph Γ. Then the maximal unitary closed variety $\mathfrak{M}' \subseteq \mathfrak{M}$ is defined by the graph $\bar{\Gamma}$, which is produced from Γ by contracting each cross section into an edge. The variety $\mathfrak{M}'$ is generated by a direct sum of semidirect products of matrix algebras. Summands correspond to connected components of the graph and factors, inside summands, – to cycles. The dimension of a matrix correspond to the length of the corresponding cycle.*

Proof. It is enough to prove the theorem for a remarkable graph. Let Γ' is produced from Γ by the contraction of edges, which don't have common verteces with cycles. We shall use the following lemmas, which are consequences of those facts: the ideal of identities of an automata algebra, defined by an irreducible cycle of order n, coincides with the ideal of identities of the matrix algebra of order n (Theorem 5.18); the ideal of identities of a semidirect product of algebras coincides with the product of their ideals of identities (Theorem 6.18, see also [83]). □

Lemma 7.48 *The T-ideal of identities of the algebra $A_{\Gamma'}$ is generated by elements of the form*

$$x_0 T(M_{n_1}) x_1 T(M_{n_2}) \dots T(M_{n_r}) x_r, \tag{11}$$

where x_i are either empty words, if the corresponding cycles are neighbor, or letters, otherwise, and by $T(M_{n_i})$ is denoted the ideal of identities of the matrix algebra M_{n_i} of order n_i. x_0 (x_r) is empty, if there is no ingoing (outgoing) arrow in the first (last) cycle. □

Lemma 7.49 *If we substitute variables in (11) by arbitrary words, with lengths not greater, than the maximum of cross section lengths of Γ, then we shall have an identity in A_Γ.* □

The theorem is a consequence of these two lemmas. □

Let us give another formulation of the theorem.

Theorem 7.50 *Each unitary closed variety of monomial algebras can be represented as a finite union of varieties of the form*

$$\mathbb{M}_{n_1} \mathbb{M}_{n_2} \dots \mathbb{M}_{n_k}.$$

Here by $\mathbb{M}_{n_i}$ is denoted the variety, generated by the matrix algebra of dimension n_i. The product of varieties is defined by the product of T-ideals (to it corresponds the semidirect product of algebras). □

Remark. Essentially, in works, dedicated to Specht problem, the generalized complexity was defined by varieties of the form $\mathbb{M}_{n_1}\mathbb{M}_{n_2}\dots\mathbb{M}_{n_k}$. If all $n_k = 1$, then we have Latyshev complexity, if $k = 1$, then we have the usual complexity.

All remaining difficulties in study of varieties of monomial algebras are connected with cross sections. The following problems seem interesting.

1. To perform an analogous study of reperesentations of monomial algebras with arbitrary graphs (including graphs with linking cycles). Which properties has Zariski closure of a representation? How can we transform a graph Γ into a more suitable form Γ', so that algebras of canonical representations of Γ and Γ' have isomorphic central extensions?
2. To construct a variety of monomial algebras, which cannot be produced by intersections, unions and products of T-ideals from nilpotent and matrix varieties.
3. To reduce the description of varieties of monomial algebras to the description of generalized varieties of monomial algebras (a part of variables, which correspond to cycles, are coefficients of generalized identities) and thus to simplify the proof of the main theorem.
4. To study the variety, generated by nilpotent monomial algebras of a bounded index, in particular, the variety, generated by a finite subword of the superword u^∞.
5. To construct an algorithm for checking the coincidence of two varieties, generated by two automata algebras.

8 Lie nilpotency: recognition and the word problem

In this chapter we study associative algebras with the Lie nilpotency identity. Algorithms, which are studied here, are related to the general theory of Groebner bases.

Let $K\langle X\rangle$ be, as above, a free associative algebra with free generators $X = \{x_1, \dots, x_n\}$ over a field K. The set X also generates the free Lie algebra $K_0\langle X\rangle^{(-)}$ over K with respect to the commutation $[y_1, y_2] = y_1y_2 - y_2y_1$. We assume that the empty word 1 belong to $\langle X\rangle$. A "long" commutator (right-normed product) is defined by the induction: $[y_1, \dots, y_r] = [[y_1, \dots, y_{r-1}], y_r]$. A long commutator can be considered, as a "nonassociative monomial". Elements of the Lie algebra $K_0\langle X\rangle^{(-)}$ will be called Lie elements of the free associative algebra $K\langle X\rangle$. Elements, which can be represented as linear combinations of long commutators, are called *eigen elements*. All Lie elements, for example, are eigen. It is well known that each totally characteristic ideal (T-ideal) is generated by eigen elements. (We assume that the characteristic of the ground field is zero.) We also assume that the reader is aquainted with the basics of the Groebner bases theory. All necessary material can be found in standart books on the computer algebra. An algebra A is called a s.f.p. (standart finitely presented) algebra, if its ideal of defining relations is generated by a Groebner base with a finite number of elements. Obviously, the solvability of the word problem in A is a consequence of this property. A property P of an algebra is called recognizable, if there exists an algorithm for checking whether an arbitrary s.f.p. algebra A has this property or not. The property of the Lie nilpotency of an algebra A means that for some r the polynomial identity $[y_1, \dots, y_r] = 0$ holds in A. The commutator ideal of such algebra is nilpotent of index $q \leq q(r, n)$, where n is the number of A generators and the function $q(r, n)$ is algorithmically computable.

Theorem 8.1 *The Lie nilpotency of a fixed index r is an algorithmically recognizable property.*

Proof. Let us consider the following types of eigen elements, which belong to the T-ideal $T^{(r)} \triangleleft K\langle X\rangle$, generated by the long commutator $[y_1, \dots, y_r]$ (as above, by q is denoted the nilpotency index of the ideal, generated by commutators in the algebra $K\langle X\rangle/T^{(r)}$):

$$\begin{array}{ll} (i) & [x_{i_1}, \dots, x_{i_r}]; \\ (ii) & [x_{i_1}, \dots, x_{i_{s_1}}] \dots [x_{j_1}, \dots, x_{j_{s_q}}]; \\ (iii) & \{f_1, \dots, f_t\} \text{ -- a base of the linear subspace in } T^{(r)}, \\ & \text{generated by eigen elements of degree } \leq D = \\ & \max\{q(r-1), r\}. \end{array} \tag{12}$$

It is easy to check, that these elements generate $T^{(r)}$, as an ideal. Hence, the algebra $A = K\langle a_1, \dots, a_n\rangle$ with n generators a_i satisfies the polynomial property $[y_1, \dots, y_r] = 0$, only when eigen polynomials (12) are zero under all substitutions $a_j \to x_i$. This is exactly the required recognition algorithm. $\square$

A variety $V^{(r)}$ of Lie nilpotent algebras of index r is finitely approximable, therefore the word problem is solvable in it. But we want to prove that this problem is solvable by an algorithm, which uses Groebner bases.

Of course, this problem is equivalent to the membership (to an ideal) problem in the free algebra of the variety $V^{(r)}$. This free algebra is the quotient algebra $A \cong K\langle X\rangle/T^{(r)}$, i.e., it is finitely presented and noncommutative polynomials (12) are its defining relations. Let $\varphi : K\langle X\rangle \to K\langle X\rangle/T^{(r)}$ be the natural epimorphism. The image $L = \varphi(K_0\langle X\rangle^{(-)}$ of the free Lie algebra is finite-dimensional in A, because all homogeneous Lie elements of degree $\geq r$ belong to the T-ideal $T^{(r)}$. If $\xi_1 = \varphi(g_1), \ldots, \xi_m = \varphi(g_m) \in L$ is a base in the Lie algebra L, $g_i \in K_0\langle X\rangle^{(-)}$, $m = dim \quad L$, then elements g_i can be chosen in a way such that $g_1 = x_1, \ldots, g_n = x_n$ and $g_{n+1}, \ldots, g_m$ are homogeneous Lie elements of degree $< r$. As $T^{(r)}$ is uniform, then g_i can be found by "linear procedures". At the same time we represented all ξ_i as some (Lie) polynomials from $\xi_1, \ldots, \xi_n$ and computed structural constants of L in this base.

Let U_L be the enveloping algebra of L and $e_1, \ldots, e_n$ be the copies of elements $\xi_1, \ldots, \xi_n$ in U_L, then e_i generate U_L. There are two canonical epimorphisms, associated with U_L. At first, the epimorphism of algebras $\psi : K\langle X\rangle \to U_L$, which extends the map of sets $\psi : X \to U_L$, $\psi(x_i) = e_i$, $i = 1, \ldots, n$. Secondly, the epimorphism of algebras $\theta : U_L \to A$, $\theta(e_i) = \xi_i$, $i = 1, \ldots, n$. The diagram of epimorphisms

$$\begin{tikzcd} k\langle X\rangle \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & U_L \arrow[d, "\theta"] \\ & A \cong k\langle X\rangle/T^{(r)} \end{tikzcd}$$

is commutative, hence, the ideal $ker\theta \quad \lhd \quad U_L$ is generated by elements $h_j(e_1, \ldots, e_n)$, where $h_j(e_1, \ldots, e_n)$ is a polynomial of the form (12). Therefore, we have the following isomorphisms

$$A \cong k\langle X\rangle/T^{(r)} \cong U_L/I.$$

Theorem 8.2 *There exists an algorithm, based on the Groebner base notion, which solves the membership to an ideal problem in the variety of Lie nilpotent algebras of a fixed index.*

Proof. We shall use the above introduced technique. Let us consider the membership to an ideal $B \lhd A$ problem, where the ideal B is generated by a finite number of elements $b_i(\xi_i, \ldots, \xi_n) = \varphi(b_i(x_1, \ldots, x_n))$, $b_i(x_1, \ldots, x_n) \in K\langle X\rangle$. This problem can be reformulated as the membership to the ideal $J \lhd U_L$ problem, where the ideal J is generated by elements $b_i(e_1, \ldots, e_n) = \psi(b_i(x_1, \ldots, x_n))$ and $h_i(e_1, \ldots, e_n)$, $I \subseteq J$. To solve the last problem, it is enough to construct a Groebner base of J and then use the reduction process with respect to this base. $\square$

Appendix

A. On rings, asimptotically close to associative

It is known that results, valid for associative PI-algebras, in many cases were found to be valid in other cases also, for alternate or Jordan rings, for example. The method of generalization uses the passage to an associative algebra of (left) multiplications. We shall formulate criterions of the asimptotic closedness to associative rings and shall present some base notions and constructions.

Definitions. An algebra A has a *bounded L-length*, if for some k its algebra $L(A)$ of left multiplications is linearly representable by elements of the form $L(p_1)\dots L(p_q)$, where $q<k$ and $L(x)$ is the operator of the left multiplication by x. A variety $\mathfrak{M}$ is called *not bad*, if the algebra of left multiplications of each f.g. algebra from $\mathfrak{M}$ is 1) finitely generated, 2) has a bounded L-length, 3) each 1-generated algebra from $\mathfrak{M}$ is associative, i.e., $\mathfrak{M}$ is a variety with *associative powers*. A variety $\mathfrak{M}$ is called *good*, if it is not bad and 4) the algebra of left multiplications of each f.g. algebra from $\mathfrak{M}$ is a PI-algebra. An algebra is called representable, if it can be embedded into a finite dimensional algebra over an associative-commutative ring.

The following statement is well known.

Proposition A.1 *a) In the category of n-dimensional representations there exists the univesal repulsing object.*

b) An algebra C is representable only when there exists a family of ideals $\{\mathcal{J}_\rangle\}_{\rangle\in\mathcal{I}}$ and a number $n\in\mathbb{N}$, such that 1) $\forall x\in C\quad \exists i\in I: x\notin \mathcal{J}_\rangle$; 2) dim $\quad C/\mathcal{J}_\rangle \le \backslash$.

Proof. . The item b) is a direct consequence of a). Let us present the construction of the universal representation, which is n-dimensional over the center. Let $\{\bar{e}_i\}_{i=1}^n$ be base vectors. The multiplication is defined by structural coefficients $C_{ij}^k : \bar{e}_i\bar{e}_j = C_{ij}^k\bar{e}_k$. To each i-th generator of C we correspond the element $\sum_j \lambda_{ij}\bar{e}_i$. Let the coefficients C_{ij}^k and λ_{ij} satisfy conditions, which are consequences of defining relation of C. The coefficients themselfs belong to the factor of a polynomial ring with respect to relations. $\square$

Notations. By Mon_r will be denoted the set of nonassociative monomials of degree r. $M^{(k)}$ is the ideal, generated by k-th powers of elements from M. If J is an ideal in C, then I_J is the ideal in $L[C]$, generated by operators of multiplication on elements from J. Let $D\subseteq C$, then by $I_{D,s}$ will be denoted the ideal in $L[C]$, generated by operators of left multiplication on elements of the form $W(t_1,\dots,t_k)$, where $W\in Mon_k$, $k\le s$ and $\exists i: t_i\in D$. If I is an ideal in $L[C]$, then $\mathcal{J}_\mathcal{I} = \{\S\in\mathcal{C} : \forall\lceil\in\rangle\lceil(\S)\quad \mathcal{L}(\lceil)\in\mathcal{I}\}$. By $g(r)$ will be denoted the number of generators in the algebra of left multiplications of the free r-generated algebra from $\mathfrak{M}$, by $l(r)$ will be denoted its length. As an algebra of left multiplications is finitely generated, then there exists a function $G(r)$, such that $I_{D,s}\supset I_{id(D)}$, where D belongs to an r-generated algebra C from $\mathfrak{M}$ and $s>G(r)$.

Proposition A.2 *If I is an ideal in $L[C]$ of codimension k, then the codimension of $\mathcal{J}_\mathcal{I}$ is $\le k\cdot N$, where N is the number of nonassociative monomials of degree $\le G(r+1)$ (r is the number of generators in C).*

Proof. As the algebra of left multiplications is finitely generated, then $x \in \mathcal{J}_{\mathcal{I}}$, if for each nonassociative monomial d of degree $\leq G(r+1)$ and, such that x occurs in d, we have that $L(d) \in I$. Therefore, the codimension of $\mathcal{J}_{\mathcal{I}}$ is not greater, than the product of codimension of I on the number of such monomials. $\square$

Theorem A.3 *a) Let $\mathfrak{M}$ be a not bad variety and $C \in \mathfrak{M}$ be a f.g. algebra. Then the representability of C is equivalent to the representability of $L[C]$.*

b) If $\mathfrak{M}$ is good and $C \in \mathfrak{M}$ is prime, then C is representable.

c) If $\mathfrak{M}$ is good and $C \in \mathfrak{M}$ is simple, then C is finite-dimensional.

d) If $\mathfrak{M}$ is good and $C \in \mathfrak{M}$ doesn't have any ideals with non-nilpotet factors, then C is simple.

e) If $\mathfrak{M}$ is good and each simple algebra in $\mathfrak{M}$ doesn't have a base, which elements are nilpotent, then the Kurosh problem has the positive solution in $\mathfrak{M}$. Moreover, if C is uniform and f.g., then there exists $M \subset C$, such that for each k the algebra $C/M^{(k)}$ is nilpotent.

Proof. The item a) is a consequence of the previous reasoning. Items c) and d) are consequences of b). The item e) is a consequence of previous propositions. Let us prove b). There exists a sequence $\{I_\alpha\}$ of ideals in $L[C]$, such that their intersection belongs to the radical $R(L[C])$ and each factor $L[C]/I_\alpha$ can be embedded into the algebra of $m \times m$ matrices (m is the same for all α). The corresponding sequence of ideals $\mathcal{J}_{\mathcal{I}_\alpha}$ in C has the following property: each factor $C/\mathcal{J}_{\mathcal{I}_\alpha}$ can be embedded into an algebra, which dimension over the center is not greater, than some m'. If $x \in \cap \mathcal{J}_{\mathcal{I}_\alpha}$, then $L(x) \in R(L[C])$ and $x \in R(C)$. But, as C is prime, then $R(C) = 0$. It remains to use Proposition A.1. $\square$

Let $q(n)$ be the degree of nilpotency of $C/M^{(n)}$ and s be the number of C generators. By $I(M,n)$ will be denoted the ideal in $L[A]$, generated by elements of the form $l(t_1)\dots l(t_n)$, such that $\exists m \in M : \forall i \quad t_i = m^{k_i}$. The following statement is a direct consequence of the definition of a not bad variety.

Proposition A.4 *a) $L[A]/I_{M^{(k)}}$ is nilpotent of degree $\leq q(k) \cdot g(s+1)$.*

b) $Id(D) \supseteq I_{D,r}$ and, if r is sufficiently big, then $Id(D) \subseteq I_{D,r}$, for all D.

c) If k is sufficiently big, then $I_{M^{(k)},r} \supseteq I(M,n)$ ($k > K(|M|,n,r)$). $\square$

Let now n will be greater, than the degree the identity, which holds in $L[A]$. Using the previous proposition and the pumping over lemma, we have the following statement.

Lemma A.5 *Let C be a f.g. graded PI-algebra from a good variety and $M \subset C$ be a finite set of uniform elements, such that the quotient algebra $C/M^{(k)}$ is nilpotent, for each k. Then there exist a number H and a finite set $D(M)$, such that the algebra $L[C]$ is linearly representable by elements of the form $t_1 t_2 \dots t_k$, where $k < H$ and either $t_i \in D$, or $\exists m_i \in M$, such that $t_i = L(x_{i1})L(x_{i2})\dots L(x_{ij})$, where all $x_{i\alpha}$ are powers of m_i.* $\square$

By the boundedness of the L-length, we have

Proposition A.6 *Let all x_α are powers of an element m. Then the product $L(x_1)L(x_2)\dots L(x_j)$ is linearly representable by elements of the form $L(y_1)L(y_2)\dots L(y_\lambda)$, where $\lambda \le l(1)$ and all y_α are powers of m.* □

Definition A.7 An algebra C has an *essential height H over a set M*, which is called an s-base of C, if there exist a finite set $D(M)$ and a number N, such that C is linearly representable by elements of the form $Q(t_1, \dots, t_l)$, where $Q \in Mon_l$, $l \le H$, and, for all i either $t_i \in D$, or $\exists m_i \in M, k_i \in \mathbb{N} : t_i = m_i^{k_i}$ and the number of $t_i \notin D$ is not greater, than H. If for some H we can take the empty set, as D, then M is a Shirshiv base of C. This is equivalent to the additional condition: M generates C, as an algebra.

Remark. The notions of the algebraicity, of the strong algebraicity and of the sparse identity can be defined analogously.

The following theorem is a consequence of Lemma A.5 and Proposition A.6.

Theorem A.8 (A.Ya.Belov) *Let C be a f.g. graded PI-algebra from a good variety and $M \subset C$ be a finite set of uniform elements. Then, if $C/M^{(k)}$ is nilpotent, for all k, then C has a bounded height over M. If M generates C, as an algebra, then M is a Shirshov base in C.* □

(We can reject the condition of the associativity of 1-generated algebras.)

Corollary A.9 *Let C be a f.g. graded PI-algebra from a good variety and $M \subset C$ be a finite set of uniform elements. Then M is an s-base, only when each simple factor of C contains a non-nilpotent image of an element from M.* □

As simple algebras from good varieties are finite-dimensional, then we have

Corollary A.10 *Let $\mathfrak{M}$ be a good variety, such that each simple algebra in $\mathfrak{M}$ doesn't have a base, which consists of nilpotent elements. Let C be a uniform f.g. algebra from $\mathfrak{M}$. Then C has a bounded height over some finite set M.* □

In several works there was proved the asimptotic closedness of some classes of algebras to associative. But, as a matter of the fact, it was proved the property of some variety to be good. In [48] the boundedness of l-length of f.g. Jordan algebras was proved, in [60] the same was proved for alternate algebras. In [60] it was proved that the algebra of left multiplications of an alternate or special Jordan f.g. PI-algebra is PI-algebra also. In the same work it was proved that the condition of the theorem holds for a f.g. alternate PI-algebra of degree m, if we take the set of words of degree $\le m^2$ for M. In [38] it was proved that the condition (4) holds for f.g. Jordan PI-algebras. In [21] it was proved the local finiteness of a Jordan PI-algebra of degree m, such that all its words of degree $\le m^2$ are algebraic. So, the following statement holds.

Corollary A.11 *a) Let A be a f.g. graded associative (alternate, Jordan) PI-algebra, $M \subset A$ be a finite set of uniform elements, which generate A, as an algebra, $M^{(k)}$ be the ideal, generated by k-th powers of elements from M. Then, if the quotient algebra $A/M^{(k)}$ is nilpotent, for each k, then A has a bounded height over M.*

b) Let B be an alternate (or Jordan) f.g. PI-algebra of degree m. Then B has a bounded height over the set of words of degree $\leq m^2$. □

The following statement holds.

Proposition A.12 *Let B be a Cayley-Dikson algebra over an arbitrary field. Then there exists a non-nilpotent word of length ≤ 2 (from B generators).* □

The following proposition is a consequence of Theorems A.8 and A.3.

Theorem A.13 *Let B be a relatively free alternate algebra and M be a set of (nonassociative) words from its generators. Then M is a Shirshov base (s-base) in B, if and only if M is a Shirshov base (s-base) of the B factor by an associative ideal.* □

Combinatorial-asimptotic notions and results can be translated to the case of good varieties. The complexity of a variety $\mathfrak{M}$ can be defined as the class of simple algebras, which belong to $\mathfrak{M}$.

Theorem A.14 *Let $\mathfrak{M}$ be a good variety. Then*

a) the essential height of representable algebra (in the case it exists) is equal to its Gelfand-Kirillov dimension and is also equal to the essential height and Gelfand-Kirillov dimension of its algebra of left multiplications;

b) for each f.g. algebra in $\mathfrak{M}$ the property of the strong algebraicity holds and also holds the natural analog of the Capelli identity. Therefore, its algebra of right multiplications is also PI;

c) analogs of Amizur and Braun theorems hold;

d) if each simple algebra in $\mathfrak{M}$ has a center, then complete analog of the theory of Razmyslov polynomials holds. In particular, the localization of a prime algebra with respect to its center is finite-dimensional over the center and a prime algebra can be embedded into an algebra, which is finite-dimensional over the center. Gelfand-Kirillov dimension is the transcendence degree of the center. □

We don't present a proof of this theorem and definitions of some notions, used in this theorem formulation, because all this is completely analogous to the associative case.

A.1 Problems and remarks

It will be interesting to obtain a description of Shirshov bases in Lie and Jordan cases. It is known that a simple Jordan algebra is either an algebra of a quadratic form, or a nonspecial algebra HC_3, or a matrix algebra with the operation $A \circ B = AB + BA$. In the first case, generators can be nilpotent, but all words of length 2 – not. In the last case (see Corollary 2.85) a set of monomials with following properties is a Shirshov base: for each regular word u of length not greater, than the dimension of matrices n, there exists a monomial from this set, such that the leading term of the element, produced after removing of parentheses, is equal to u. Hence, to improve the estimations in Zelmanov result [21], it is enough to perform computations in the algebra HC_3. In each case the estimation on degree of words will be of order $m/2 + const$. Let us note that the above condition on a set of Jordan (Lie) monomials is sufficient, but not necessary. Therefore, the problem arises about the description of sets of monomials, which are Shirshov bases. Here it is enough to check only the condition of Theorem A.13.

It is known that all simple PI(λ, δ)-algebras are associative. So, the problem arises: is the variety of PI(λ, δ)-algebras good?

Obviously, a f.g. Engel Lie algebra generates a not bad variety. Is it possible to give a direct proof that it is good? Then we shall have another proof of its nilpotency (a well known result by E.I.Zelmanov).

Is an algebra, which good from the left, good from the right also? Is it true in the case of zero characteristic?

Is the class of not bad (good) varieties in the f.g. case closed with respect to tensor products? And in general case?

One of conditions of our proof of Kurosh problem is the absence of a nil-base in a simple algebra, This condition holds in associative, Jordan and alternate cases. On the other hand, a finite Lie algebra generates a good variety with a nil-base. It will be interesting to obtain general criterions of the absence of a nil-base. Is it possible to change the condition in Corollary A.10 to a more weak one: the absence of simple nil-algebras?

B. On Nagata-Higman theorem for semirings

By a semiring will be called a set with two operations: the addition and the multiplication. The addition is commutative and is connected with the multiplication by the distributive low. The semiring specific character is in the absence of subtraction and in the existence of the kernel of the addition. We can also consider semirings – algebras over an associative-commutative semiring Φ. Each semiring is an algebra over $\mathbb{N}$. We shall study only associative semirings with zero.

If an identity of the form

$$x_1 \dots x_m = \sum_{\sigma \neq \mathrm{id}} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(m)}, \qquad \text{where } \alpha_\sigma \in \Phi, \tag{13}$$

holds in a semiring, then the height theorem also holds in it. However, unlike the ring case, here the validity of even the polylinear identity doesn't imply the validity of an identity of the form (13). Let us consider, as an example, a semiring, generated by generators of a relatively free algebra. Then we have

Proposition B.1 *Let A be a relatively free non-nilpotent algebra, $\mathfrak{M} = \mathrm{Var}(A)$, $a_1, \dots, a_s$ be generators of A and R be the semiring, generated by a_i. Then a word v in R is a linear combination of other words, only when $v = w$, where w is a different word, and then this linear combination consists of one term w.*

Proof. In a relatively free algebra each relation is an identity and, by the non-nilpotency of A, the sum of coefficients of all terms in identity is zero. On the other hand, all coefficients of a linear combination are positive integers. $\square$

If we take several generic matrices of dimension 2×2 or bigger, then they generate an absolutely free semigroup, therefore the above equality is impossible, if $\mathfrak{M}$ has the complexity > 1 and A is considered over an infinite field (i.e., $\mathfrak{M}$ has the complexity > 1, in the sense of Section 2.2.8). We have the following proposition.

Corollary B.2 *If A satisfies the conditions of the previous proposition and $PIdeg(\mathfrak{M}) > 1$ (in the sense of the definition from Section 2.2.8), then words in R cannot be represented by a linear combinations of other words. In particular, the height theorem doesn't hold in R.* $\square$

Let an identity of the form $f = 0$ holds in a semiring S. The "partial linearization" of f doesn't hold in R. (For example, in the variety, generated by the identity $x^3 = 0$, the identity $x^2y + xyx + xy^2 = 0$ doesn't hold.) But the complete linearization of f with respect to each variable holds.

Proposition B.3 *Let the identity $f(t, x) = 0$ holds in a semiring S. (Here by t is denoted a system of variables, which are different from x.) Let f has degree m with respect to x. Then $\tilde{f}$ holds in S, where $\tilde{f}$ is the complete linearization of f with respect to x.*

Proof. In the ring case we can write the equality in the following way

$$\tilde{f} = f(t, x_1 + \dots + x_m) - \sum_i f(t, x_1 + \dots + \widehat{x}_i + \dots + x_m) +$$

$$+ \sum_{i<j} f(t, x_1 + \dots + \widehat{x}_i + \dots + \widehat{x}_j + \dots + x_m) - \dots + (-1)^{m-1} \sum_k f(t, x_k).$$

(The sign $\widehat{\ }$ means, that the corresponding term is deleted.)

But there are no subtraction in semirings, therefore we have to write the equality in the following way

$$\tilde{f} + \sum_i f(t, x_1 + \dots + \widehat{x}_i + \dots + x_m) +$$
$$+ \sum_{i<j<k} f(t, x_1 + \dots + \widehat{x}_i + \dots + \widehat{x}_j + \dots + x_m) + \ldots =$$
$$= f(t, x_1 + \dots + x_m) + \sum_{i<j} f(t, x_1 + \dots + \widehat{x}_i + \dots + \widehat{x}_j + \dots + x_m) + \ldots .$$

It remains to notice that, if f is an identity, then all terms in this equality, except, maybe $\tilde{f}$, are zero, hence $\tilde{f}$ is zero too. □

Corollary B.4 *The identity $\sum_\sigma x_{\sigma(1)} \ldots x_{\sigma(m)} = 0$ in semirings is a consequence of the identity $x^m = 0$.* □

Remark. a) Let us note that in the proof of the above proposition we didn't use the associativity.

b) As there are no subtraction, then in the semiring case we work with the left hand part and the right hand part of an equality separately. The equality $x = y$, for example, can be proved in the following way: we prove the equality $x + z = y + t$ and then prove that $z = t = 0$ (the equality to zero is essential!).

Now we are ready to prove Nagata-Higman theorem for semirings.

Theorem B.5 *Let A be an l-generated semiring with the identity $x^m = 0$. Then A is nilpotent of degree $\leq 2l^{m+1}m^3$.*

Proof. Let us prove that all words of this length are zero. Let us suppose the contrary and let v be the lexicographically minimal nonzero word of length $2l^{m+1}m^3$. By results of Section 2, each such word is either m-divided, or contains an m-th power of a subword. Hence, v is m-divided and $v = v_0v_1v_2 \ldots v_m$, where $v_1 \succ v_2 \succ \dots \succ v_m$. Then, for each $\sigma \in S_m \setminus \mathrm{id}$, the word $v_\sigma = v_0v_{\sigma(1)} \ldots v_{\sigma(m)}$ is lexicographically smaller, than v, and, by the v minimality, is equal to zero. So

$$v = v_0v_1v_2 \ldots v_m = v_0(v_1v_2 \ldots v_m + \sum_{\sigma \neq \mathrm{id}} v_{\sigma(1)} \ldots v_{\sigma(m)}) =$$
$$= v_0(\sum_\sigma v_{\sigma(1)} \ldots v_{\sigma(m)}) = v_0 \cdot 0 = 0.$$

We have a contradiction to the choice of v. Nagata-Higman theorem for semirings is proved. □

Remark. It will be interesting to generalize this theorem on a suitable class of nonassociative semirings (Jordan or alternate, for example).

The case of semirings with a noncommutative addition If R is a ring, then, by the distributivity, $(x + y - x - y)z = (x + y)(z - z) = 0$. Hence, $(x+y)z = (y+x)z$ and the noncommutativity of the addition is not interesting. But, we cannot perform such computations in semirings. Moreover, in the case of a noncommutative addition there is no linearization. The following proposition allows to bypass this difficulty.

Proposition B.6 *Let $x+y = y+x' = 0$, then, for all a and b, $ax+bx = bx+ax$ and $xa + xb = xb + xa$.*

Proof. $0 = (a + b)(x + y) = ax + bx + ay + by$. On the other hand, $bx + ax + ay + by = bx + a(x + y) + by = b(x + y) = 0$. Therefore, $bx + ax + (a + b)y = ax + bx + (a + b)y$. By adding $(a + b)x'$ to the righthand side, we have the required equality. The equality $xa + xb = xb + xa$ can be proved analogously. □

In the case of a commutative addition, by this proposition and Nagata-Higman theorem, we have

Corollary B.7 *If W is a word of length $2l^{m+1}m^3$ and $x+y = y+x' = 0$, then $Wx = xW = Wx' = x'W = 0$.* □

Let the identity $x^m = 0$ holds. Let us consider the expressions $(x_1 + \cdots + x_m)^m$ and $(x_m + \cdots + x_1)^m$. If we remove the parentheses (at first, in the first factor, then in the second, and so on), then we shall get two sums $\sum_{i=1}^{m^m} v_i$ and $\sum_{i=1}^{m^m} v_i'$, each of m^m terms – all possible products of x_i, and $v_i' = v_{m^m - i}$. (If we rewrite all terms in one sum in the reverse order, then we shall get the another sum.) Let $s_k = \sum_{i=1}^{k-1} v_i$, $s_k' = \sum_{i=m^m-(k-1)}^{m^m} v_i'$, $r_k = \sum^{m^m} i = kv_i$, $r_k' = \sum_{i=1}^{m^m-(k-1)} v_i'$. Then $s_k + r_k = r_k' + s_k' = 0$, $s_k + v_k = s_{k+1}$, $v_{k-1} + r_k = r_{k-1}$, $v_k + s_k' = s_{k+1}'$, $r_k' + v_{k-1} = r_{k-1}'$.

Lemma B.8 *Let the length of a word W is $\geq k \cdot 2l^{m+1}m^3$. Then $Ws_k = Wv_k = Wr_k = Ws_k' = Wr_k' = 0$. the same is true for the right multiplication of W by s_k, r_k, s_k', r_k' and v_k.*

Proof. By the symmetry, it is enough to check the equalities $Ws_k = Wr_k = Wv_k = 0$. Let us use the induction on k. Corollary B.7 proves the statement for $k = 1$. By $s_k + r_k = 0$, it is enough to check the equality $Wr_k = 0$. Let $W = uW'$, where $|u| = 2l^{m+1}m^3$, $|W'| \geq (k-1) \cdot 2l^{m+1}m^3$. By the inductive supposition, $W's_{k-1} = W'v_{k-1} = W'r_{k-1} = W's_{k-1}' = W'r_{k-1}' = 0$. Hence, $W's_k = W's_k' = W'v_k$ and $W'v_k + W'r_k = W's_k + W'r_k = W'(s_k + r_k) = 0$. Analogously, $W'r_k' + W's_k' = W'r_k' + W'v_k = W'r_k' + W's_k = 0$. It remains to set $x = W'r_k, y = W's_k, x' = W'r_k'$ and to use the previous proposition. □

Corollary B.9 (Nagata-Higman theorem for general semirings.) *If the identity $x^m = 0$ holds in an l-generated semiring, then each word with length $> m^m \cdot 2l^{m+1}m^3 + m$ is zero.*

Proof. The index k changes from 1 to m^m. So, the product of a word of length $m^m \cdot 2l^{m+1}m^3$ on any term of any sum $\sum_{i=1}^{m^m} v_i$ is zero. □

Remark. a) Using powers of partial sums and symmetry considerations, it is easy to improve the estimation up to $m! \cdot l^{m+1}m^3 + m$. It will be interesting to get an exponential (with respect to m) estimation.

b) We used the existence of zero. Otherwise, the addition has the kernel and our reasoning fails. There exists a non-locally finite semiring with the identity $x^2 = x^3$ (it can be constructed with the help of a squarefree superword from three letters).

c) We used the bilateral distributivity. Otherwise, even in the case of a commutative addition, a product of sums cannot be transformed into a sum of products, and our reasoning fails. Probably, in the case of near-rings (rings with the one-sided distributivity) Nagata-Higman theorem is wrong.

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