1 Definitions

Définition 1. natural transformation

Définition 2. Let $\mathcal{C}$ be a category.

A monad on the category $\mathcal{C}$ is (T, μ, η) where $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor and where $\mu : TT \to T$ and $\eta : \mathrm{id}_{\mathcal{C}} \to T$ are natural transformations such that the following diagrams commute :

$$\begin{tikzcd}
T \circ T \circ T \arrow[r, "T \circ \mu"] \arrow[d, "\mu \circ T"] & T \circ T \arrow[d, "\mu"] \\
T \circ T \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "\eta \circ T"] \arrow[d, "T \circ \eta"] \arrow[dr, "id"] & T \circ T \arrow[d, "\mu"] \\
T \circ T \arrow[r, "\mu"] & T
\end{tikzcd}$$

Définition 3. A monoidal category is $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ where

- $\mathcal{C}$ is a category,
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor of $\mathcal{C}$,
- I is an object of $\mathcal{C}$,
- λ and ρ are invertible natural transformations given for every objet A of $\mathcal{C}$ by
$$\lambda_A : I \otimes A \to A \qquad \rho_A : A \otimes I \to A,$$
- α is an invertible natural transformation given for every objects A, B and C of $\mathcal{C}$ by
$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

such that the following diagrams commute :

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A, B \otimes C, D}"] & A \otimes ((B \otimes C) \otimes D \arrow[d, "A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes B"'] & & A \otimes (I \otimes B) \arrow[dl, "A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

Définition 4. A monoidal strict category $(\mathcal{C}, \otimes, I)$ is a monoidal category $(\mathcal{C}, \otimes, I, \mathrm{id}_\mathcal{C}, \mathrm{id}_\mathcal{C}, \mathrm{id}_\mathcal{C})$.

monoid in $(Cat, \times, 1)$

Définition 5. A strong monad (T, η, μ, t) on a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ is a monad (T, η, μ) on the category $\mathcal{C}$ equipped with a natural transformation t called right-force, defined by

$$t_{A,B} : A \otimes TB \to T(A \otimes B)$$

such that the following diagrams commute for any objects A, B and C of $\mathcal{C}$:

$$\begin{tikzcd}
I \otimes TA \arrow[r, "t_{I,A}"] \arrow[dr, "\lambda_{TA}"'] & T(I \otimes A) \arrow[d, "T\lambda_A"] \\
& TA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes \eta_B"] \arrow[dr, "\eta_{A \otimes B}"'] & A \otimes TB \arrow[d, "t_{A,B}"] \\
& T(A \otimes B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes TC \arrow[rr, "t_{A \otimes B, C}"] \arrow[d, "\alpha_{A,B,TC}"'] & & T((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \otimes (B \otimes C) \arrow[r, "A \otimes t_{B,C}"'] & A \otimes T(B \otimes C) \arrow[r, "t_{A, B \otimes C}"'] & T(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes TTB \arrow[r, "t_{A,TB}"] \arrow[d, "A \otimes \mu_B"'] & T(A \otimes TB) \arrow[r, "Tt_{A,B}"] & TT(A \otimes B) \arrow[d, "\mu_{A,B}"] \\
A \otimes TB \arrow[rr, "t_{A,B}"'] & & T(A \otimes B)
\end{tikzcd}$$

The notion of left-force on a monad is defined similarly.

Définition 6. A symmetric monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha, \gamma)$ is a monoidal category $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha)$ equipped with a natural transformation γ defined for all objects A and B of $\mathcal{C}$ by

$$\gamma_{A,B} : A \otimes B \to B \otimes A$$

such that the following diagrams commute

$$\begin{tikzcd}
B \otimes A \arrow[r, "\gamma_{B,A}"] \arrow[dr, "id_{B \otimes A}"'] & A \otimes B \arrow[d, "\gamma_{A,B}"] \\
& B \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B} \otimes C"'] & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "B \otimes \gamma_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

Lemme 7. Let $(\mathcal{C}, \otimes, I, \lambda, \rho, \alpha, \gamma)$ a symmetric monoidal category with a monad (T, η, μ).

Every left-force on the monad induces a right-force on the monad. Conversely, every right-force on the monad induces a left-force on the monad.

Proof. Let t be a left-force. The natural transformation t induces a natural transformation s defined by :

$$s_{A,B} = T\gamma_{A,B} \circ t_{B,A} \circ \gamma_{TA,B} : TA \otimes B \to T(A \otimes B)$$

Let us now show that s is indeed a right-force.

$$\begin{tikzcd}
TA \otimes I \arrow[r] \arrow[drr] & I \otimes TA \arrow[dr] \arrow[d, phantom, "(1)"] & T(I \otimes A) & T(A \otimes I) \\
 & {} & TA &
\end{tikzcd}$$

$\square$

2 Forests and monads

Définition 8. A forest is given by a set V and a mapping

$$p : V \to V \uplus \{\perp\}$$

such that