DE LA RECHERCHE À L'INDUSTRIE

# Presenting Categories Modulo a Rewriting System

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# Rewriting systems and monoids

Let us consider a rewriting system $(P_1, P_2)$ :

- $P_1$ the alphabet $x, y \ldots$,
- $P_2 \subset P_1^* \times P_1^*$ a set of rules $w_1 \rightarrow w_2$.

This rewriting system $P = (P_1, P_2)$ presents a monoid :

- $P_1^*$ is the free monoid generated by $P_1$,
- call $\overset{*}{\leftrightarrow}$ the congruence generated by the rules in $P_2$ (symmetric, reflexive, transitive and context closure of $\rightarrow$),
- the rewriting system $P$ presents the monoid $\|P\| = P_1^* / \overset{*}{\leftrightarrow}$
- when the rewriting system is convergent, the elements of $\|P\|$ are isomorphic to normal forms wrt $P$

# Example : Presentation of $\mathbb{N}$

Let's try to present the monoid $\mathbb{N}$ :

- We need a generator $a$
- The elements of the free monoids are $a^0, a^1, a^2, a^3, \ldots$
- There is an isomorphism of monoids between the generated monoid and $\mathbb{N}$

We get the presentation $(P_1, P_2)$ with :

$$P_1 = \{a\}$$
$$P_2 = \emptyset$$

# Example : Presentation of $\mathbb{N} \times \mathbb{N}$

- It's two copies of $\mathbb{N}$, so let's take one generator for each copy : $a = (1, 0) \qquad b = (0, 1)$
- The operation is given by the sum componentwise : $(i, j) \otimes (k, l) = (i + k, j + l)$
- The elements of the free monoids are words with $a$ and $b$

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  $(i, j) \otimes (k, l) = (i + k, j + l)$
- The elements of the free monoids are words with $a$ and $b$
- There are two many of them
- We need the rule $ba \to ab$
- The normal forms are $a^m b^n$ with $(m, n)$ in $\mathbb{N} \times \mathbb{N}$
- The presented monoid is isomorphic to $\mathbb{N} \times \mathbb{N}$

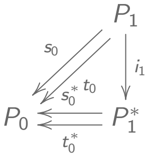
We get the presentation $(P_1, P_2)$ with

$$P_1 = \{a, b\}$$
$$P_2 = \{ba \to ab\}$$

# Presentation of category

This is a generalization of the presentation of monoids (a monoid is a category with one object)

- Consider a graph $(P_0, P_1)$ (source and target functions $s_0, t_0$),
- It generates a free category with objects $P_0$ and morphisms $P_1^*$



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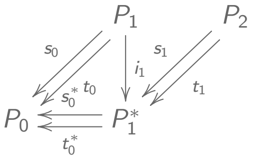
- It generates a free category with objects $P_0$ and morphisms $P_1^*$
- Consider a set $P_2 \subset P_1^* \times P_1^*$ of relations $\alpha : f \Rightarrow g$ such that

$$s_0^* \circ s_1 = s_0^* \circ t_1 \qquad t_0^* \circ s_1 = t_0^* \circ t_1$$

  with $s_1(\alpha) = f$ and $t_1(\alpha) = g$
- A relation $\alpha$ is a rewriting rule $f \rightarrow g$ (with $f$ and $g$ parallel)



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- Consider a graph $(P_0, P_1)$ (source and target functions $s_0, t_0$),
- It generates a free category with objects $P_0$ and morphisms $P_1^*$
- Consider a set of relations $P_2 \subset P_1^* \times P_1^*$
- Similarly to the case of monoids, we consider the normal forms of $P_1^*$ wrt this rewriting system. If the rewriting system is convergent, they are equivalent to the equivalence classes wrt $\overset{*}{\leftrightarrow}$
- The category presented is the category $\|P\|$ with objects $P_0$ and morphisms $P_1^* / \overset{*}{\leftrightarrow}$

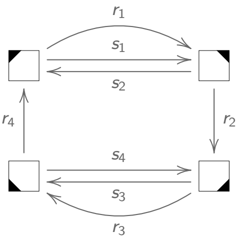
# Example : the dihedral category (1/2)

## Definition

The *dihedral group* $D_n$ is the group of isometries of the plane preserving a regular polygon with $n$ faces. This group is generated by a rotation $r$ of angle $2\pi/n$ and a reflection $s$, and can be described as the free group over the two generators $r$ and $s$ quotiented by the congruence generated by the three relations $s^2 = \mathrm{id}$, $r^n = \mathrm{id}$ and $rsr = s$.

# Example : the dihedral category $D_4^\bullet(2/2)$

We consider a variant where a vertex of the square is distinguished :



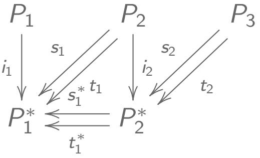

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \qquad s_{j+1} \circ s_j = \mathrm{id} \qquad r_j \circ s_{j+1} \circ r_j = s_j$$

$$s_j \circ s_{j+1} = \mathrm{id} \qquad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

for $i \in \{1, \ldots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \ldots, 4\}$.

# Presentation of a monoidal category

- Consider a set of generators for objects $P_1$
- Generate the free monoid $P_1^*$
- Consider a set of generators for morphisms $P_2$ between objects in $P_1^*$
- Consider the free monoidal category with objects $P_1^*$ and morphisms $P_2^*$
- Consider a set of relations $P_3 \subset P_2^* \times P_2^*$. Its elements are rewriting rules $f \Rightarrow g$.
- If the rewriting system is convergent, the equivalence classes of $P_2^*$ modulo $\overset{*}{\Leftrightarrow}$ are isomorphic to normal forms of elements of $P_2^*$ wrt $P_3$
- The monoidal category presented by $P = (P_1, P_2, P_3)$ is the category with objects $P_1^*$ and with morphisms $P_2^* / \overset{*}{\Leftrightarrow}$



# An example : $\Delta$

$\Delta$ is the monoidal category

- with objects : $\mathbb{N}$
- morphisms $m \to n$ : increasing functions $[m] \to [n]$ with $[m] = \{0, \ldots, m-1\}$
- tensor product : $m \otimes n = m + n$

# An example : $\Delta$

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- tensor product : $m \otimes n = m + n$

What is a presentation $(P_1, P_2, P_3)$ of $\Delta$ ?

$$P_1^* = \{a^m\}\ m \in \mathbb{N}$$

$$\begin{aligned} P_1 &= \{a\} \\ P_2 &= \{\mu : a^2 \to a, \eta : 1 \to a\} \\ P_3 &= \{A : \mu \circ (\mathrm{id} \otimes \mu) \Rightarrow \mu \circ (\mu \otimes \mathrm{id}), \\ &\qquad R : \mu \circ (\mathrm{id} \otimes \eta) \Rightarrow \mathrm{id}, \\ &\qquad L : \mu \circ (\eta \otimes \mathrm{id}) \Rightarrow \mathrm{id}\} \end{aligned}$$

# Normal forms

$$
\begin{aligned}
P_1 &= \{a\} \\
P_2 &= \{\mu : a^2 \to a, \eta : 1 \to a\} \\
P_3 &= \{A : \mu \circ (\mathrm{id} \otimes \mu) \Rightarrow \mu \circ (\mu \otimes \mathrm{id}), \\
&\quad R : \mu \circ (\mathrm{id} \otimes \eta) \Rightarrow \mathrm{id}, \\
&\quad L : \mu \circ (\eta \otimes \mathrm{id}) \Rightarrow \mathrm{id}\}
\end{aligned}
$$

The normal forms in $P_2^*$ are tensor products of :

$$\eta \qquad \mu \circ (\mathrm{id} \otimes (\mu \circ (\mathrm{id} \otimes (\ldots))))$$

# Beware

## The problem with presentations of monoidal category

It is required that the underlying monoid is free !

# Intuition of the problem : $\Delta \times \Delta$

What is a presentation of $\Delta \times \Delta$ ?
The underlying monoid $\mathbb{N} \times \mathbb{N}$ is not free but let's try :

$$P_1 = \{a, b\}$$
$$P_2 = \{\mu_a : a^2 \to a, \eta_a : 1 \to a, \mu_b : b^2 \to b, \eta_b : 1 \to b\}$$
$$P_3 = \{A_a, R_a, L_a, A_b, R_b, L_b\}$$

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The underlying monoid $\mathbb{N} \times \mathbb{N}$ is not free but let's try :

$$P_1 = \{a, b\}$$
$$P_2 = \{\mu_a : a^2 \to a, \eta_a : 1 \to a, \mu_b : a^2 \to a, \eta_b : 1 \to a, \gamma : ba \to ab\}$$
$$P_3 = \{A_a, R_a, L_a, A_b, R_b, L_b, ...?\}$$

The set of objects is too big : we would need to still have the rewriting rule $ba \to ab$.
This morphism $\gamma$ is what we call an *equational morphism* : we would like to be able to consider the objects modulo this relation between objects

# Main Result

Set of equational morphisms $\tilde{P}_2$

Three constructions :

- $\|P\|{\downarrow}\tilde{P}_2$ Consider normal forms wrt to $\tilde{P}_2$ for objects
- $\|P\|/\tilde{P}_2$ Consider equational morphisms as identities: quotient
- $P[\tilde{P}_2^{-1}]$ Consider equational morphisms as reversible: localization

## Theorem

*If the presentation $(P, \tilde{P}_2)$ admits good properties, then*

$\|P\|{\downarrow}\tilde{P}_2 \xrightarrow{\textit{isomorphism}} \|P\|/\tilde{P}_2$

$\|P\|{\downarrow}\tilde{P}_2 \xrightarrow{\textit{equivalence}} P[\tilde{P}_2^{-1}]$

$\|P\|/\tilde{P}_2 \xrightarrow{\textit{equivalence}} P[\tilde{P}_2^{-1}]$

# Presentation modulo

for simplicity, restrict to categories

## Definition

A presentation modulo is $(P, \tilde{P}_1)$ where

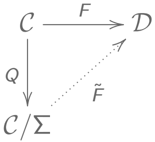
- $P$ a presentation of category,
- $\tilde{P}_1$ a subset of $P_1$ (set of equational morphisms).

# Quotient $\mathcal{C}/\Sigma$

intuition : we identify the equational morphisms with identities

## Definition

The *quotient* of a category $\mathcal{C}$ by a set $\Sigma$ of morphisms of $\mathcal{C}$ is a category $\mathcal{C}/\Sigma$ together with a *quotient functor* $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ sending the elements of $\Sigma$ to identities, such that for every functor $F : \mathcal{C} \to \mathcal{D}$ sending the elements of $\Sigma$ to identities, there exists a unique functor $\tilde{F}$ such that $\tilde{F} \circ Q = F$.

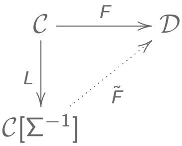


# Localization $\mathcal{C}[\Sigma^{-1}]$

intuition : we identify the equational morphisms with isomorphisms

## Definition

The *localization* of a category $\mathcal{C}$ by a set $\Sigma$ of morphisms is the category $\mathcal{C}[\Sigma^{-1}]$ together with a *localization functor* $L : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ sending the elements of $\Sigma$ to isomorphisms, such that for every functor $F : \mathcal{C} \to \mathcal{D}$ sending the elements of $\Sigma$ to isomorphisms, there exists a unique functor $\tilde{F}$ such that $\tilde{F} \circ L = F$.



# Explicit constructions

## Lemma

*The presentation* $(P_0, P_1', P_2')$ *where* $P_1' = P_1 \uplus \left\{ \overline{f} : y \to x \;\middle|\; f : x \to y \in \tilde{P}_1 \right\}$ *and where* $P_2' = P_2 \uplus \left\{ \overline{f} \circ f \Rightarrow \mathrm{id}, f \circ \overline{f} \Rightarrow \mathrm{id} \;\middle|\; f \in \tilde{P}_1 \right\}$ *presents the localization of the category* $\|P\|$ *by* $\tilde{P}_1$.

## Lemma

*The presentation* $(P_0', P_1', P_2')$ *where*

- $P_0' = P_0 / \cong_1$ *where* $\cong_1$ *is the smallest equivalence such that* $x \cong_1 y$ *whenever there exists a generator* $f : x \to y$ *in* $\tilde{P}_1$, *and we denote* $[x]$ *the equivalence class of* $x \in P_0$,
- *the elements of* $P_1'$ *are* $f : [x] \to [y]$ *for* $f : x \to y$ *in* $P_1$,
- *the elements of* $P_2'$ *are of the form* $\alpha : f \to g$ *for* $\alpha : f \to g$ *in* $P_2$, *or* $\alpha_f : f \to \mathrm{id}_{[x]}$ *for* $f : x \to y$ *in* $\tilde{P}_1$,

*presents the quotient category* $\|P\| / \tilde{P}_1$.

# Category of normal forms $\|P\|\downarrow\tilde{P}_1$

idea : chose a representative of the equivalence classes

## Assumption

The rewriting system on $P_0$ with rules given by $\tilde{P}_1$ is convergent.

## Definition

The *category of normal forms* is full subcategory of $\|P\|$ whose objects are the normal forms in $P_0$ wrt $\tilde{P}_1$

# Main Result

## Theorem

*If the presentation $(P, \tilde{P}_1)$ admits good properties, then*

$$\begin{array}{ccc} \|P\| \downarrow \tilde{P}_1 & \xrightarrow{\textit{isomorphism}} & \|P\| \,/\, \tilde{P}_1 \\ & \searrow^{\textit{equivalence}} \quad {}^{\textit{equivalence}}\swarrow & \\ & P[\tilde{P}_1^{-1}] & \end{array}$$

# Why this question?

Consider the category

$$\mathcal{C} = \; x \overset{f}{\underset{g}{\rightrightarrows}} y$$

Let's consider $\tilde{P}_1 = \{f, g\}$.
The quotient is

$$\overline{x} \circlearrowright \mathrm{id}$$

The localization is equivalent to

$$\star \circlearrowright m \in \mathbb{Z}$$

Quotient and localization are not equivalent !

# Why this question?

Consider the category

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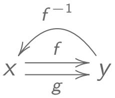
The category of normal forms is


The localization is generated by




with $f$ and $f^{-1}$ inverses
Normal forms and localization are not equivalent !

# Why this question?

Consider the category

$$\mathcal{C} = \; x \overset{f}{\underset{g}{\rightrightarrows}} y$$

Let's consider $\tilde{P}_1 = \{f\}$.
The category of normal forms is

$$y \circlearrowleft \mathrm{id}$$

The quotient is

$$\overline{y} \circlearrowleft g$$

Normal forms and quotient are not isomorphic !

# Assumption 1 : convergence

## Assumption

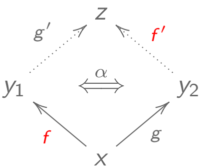
The rewriting system on $P_0$ with rules $\tilde{P}_1$ is convergent (terminating and confluent).

# Assumption 2 : residuation

Intuition : an equational should not change anything.

## Assumption

for every pair of distinct coinitial generators $f : x \to y_1$ in $\tilde{P}_1$ and $g : x \to y_2$ in $P_1$, there exist a pair of cofinal morphisms $g' : y_1 \to z$ in $P_1^*$ and $f' : y_2 \to z$ in $\tilde{P}_1^*$ and relation $\alpha : g' \circ f \Leftrightarrow f' \circ g$ in $P_2$,



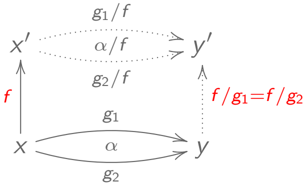

We suppose fixed an arbitrary choice of a particular triple $(g', f', \alpha)$ associated to it, and write $g/f$ for $g'$ *residual of* $g$ *after* $f$, $f/g$ for $f'$ *residual of* $f$ *after* $g$.

# Assumption 3 : Cylinder property

Intuition : an equational should not change anything.

## Assumption

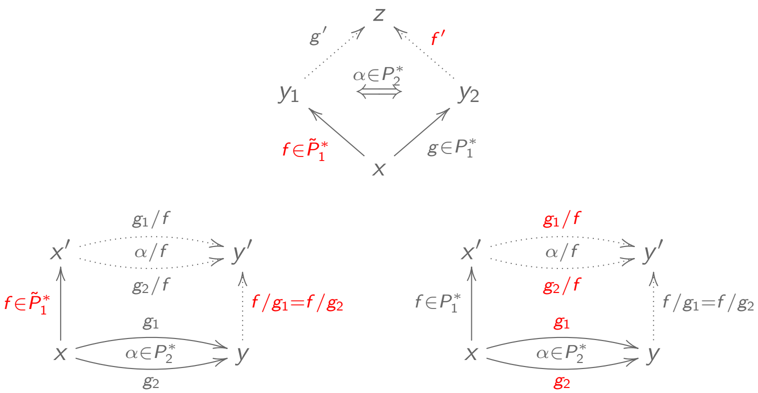
for every triple of coinitial morphisms $f : x \to x'$ in $\tilde{P}_1$ (resp. in $P_1$) and $g_1, g_2 : x \to y$ in $P_1^*$ (resp. in $\tilde{P}_1^*$) such that there exists a relation $\alpha : g_1 \Leftrightarrow g_2$, we have $f/g_1 = f/g_2$ and there exists a 2-cell $g_1/f \overset{*}{\Leftrightarrow} g_2/f$. We write $\alpha/f$ for an arbitrary choice of such a 2-cell.



# Assumption 4 : termination properties

They ensure that we get global residuation (on morphisms) and global cylinder :

## Theorem



# Assumption 5 : opposite

## Assumption

The *opposite presentation modulo* $(P^{\mathrm{op}}, \tilde{P}_1^{\mathrm{op}})$ with

- $P^{\mathrm{op}} = (P_0, P_1^{\mathrm{op}}, P_2^{\mathrm{op}})$
- $P_1^{\mathrm{op}} = \{f^{\mathrm{op}} : y \to x \mid f : x \to y \in P_1\}$
- $P_2^{\mathrm{op}} = \{\alpha^{\mathrm{op}} : f^{\mathrm{op}} \Rightarrow g^{\mathrm{op}} \mid \alpha : f \Rightarrow g\}$ with $f^{\mathrm{op}} = f_1^{\mathrm{op}} \circ ... \circ f_k^{\mathrm{op}}$ for $f = f_k \circ \ldots \circ f_1$
- $\tilde{P}_1^{\mathrm{op}}$ is the subset of $P_1^{\mathrm{op}}$ corresponding to $\tilde{P}_1$.

also satisfies previous assumptions

# Isomorphism between $\|P\| /\tilde{P}_1$ and $\|P\|\downarrow\tilde{P}_1$

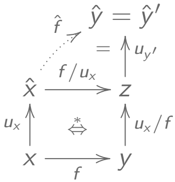
**Theorem**

*The quotient of $\|P\|$ by $\tilde{P}_1$ is isomorphic to the category of normal forms $\|P\|\downarrow\tilde{P}_1$.*

# Sketch of proof (1/2)

- construct quotient functor $N : \|P\| \to \|P\|\!\downarrow\!\tilde{P}_1$ by

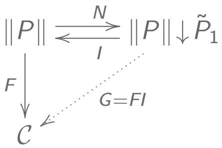
$$Nx = \hat{x} \text{ its normal form}$$
$$Nf = \hat{f}$$




- the functor $N$ is correctly defined thanks to the global properties on the residuals and of the cylinder
- $N\tilde{P}_1$ is a subset of the identities

# Sketch of proof (2/2)

We define the inclusion functor $I : \|P\| \downarrow \tilde{P}_1 \to \|P\|$




- $F = GN$ checked on objects and morphisms
- uniqueness of $G$ comes from the fact that $NI = \mathrm{Id}_{\|P\| \downarrow \tilde{P}_1}$

# Faithfulness of $L$

## Theorem

*The localization functor* $L : \|P\| \to \|P\|\,[\tilde{P}_1^{-1}]$ *is faithful.*

## Remark

This is a generalization of a similar result of Dehornoy on embedding a presented monoid with good properties into its envelopping groupoid

# Sketch of proof of the faithfulness of $L$

Relies on the following proposition :

## Proposition

If the equational morphisms are epimorphisms and monomorphisms of $\|P\|$, then the localization functor $L$ is faithful

- From the global cylinder, we get that the equational morphisms are epi : consider residual after $f$ equational morphism of

$$g_1 \circ f \overset{*}{\Leftrightarrow} g_2 \circ f$$

- From the same assumptions on the opposite presentation, we get that the equational morphisms are mono

# Equivalence between $\|P\|\,[\tilde{P}_1^{-1}]$ and $\|P\|\downarrow\tilde{P}_1$

**Theorem**

*The categories $\|P\|\,[\tilde{P}_1^{-1}]$ and $\|P\|\downarrow\tilde{P}_1$ are equivalent.*

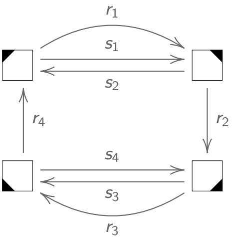
# Sketch of proof of equivalence

- Construction of the equivalence functor

$$S : \|P\|\downarrow\tilde{P}_1 \xrightarrow{I} \|P\| \xrightarrow{L} \|P\|\,[\tilde{P}_1^{-1}]$$

- $S$ is full : using calculus of fractions, for $(f, u)$ in $\|P\|\downarrow\tilde{P}_1(\hat{x}, \hat{y})$, we get that $u = \mathrm{id}$ and $Ff = (f, u)$
- faithfulness : $I$ is faithful, $L$ is faithful

# Our example : the dihedral category $D_4^\bullet$




- This presentation does not satisfy the assumptions

# Tietze-transformation

## Definition

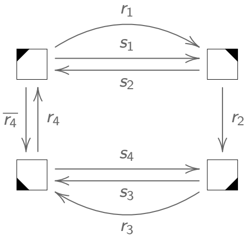
Given a presentation $P$, a *Tietze transformation* consists in

- adding (resp. removing) a generator $f \in P_1$ and a relation $\alpha : f \Rightarrow g \in P_2$ with $g \in (P_1 \setminus \{f\})^*$,
- adding (resp. removing) a relation $\alpha : f \Rightarrow g \in P_2$ such that $f$ and $g$ are equivalent wrt the congruence generated by the relations in $P_2 \setminus \{\alpha\}$.

## Proposition

Two presentations $P$ and $P'$ are related by a finite sequence of Tietze transformations if and only if they present the same category, i.e. $\|P\| \cong \|P'\|$.

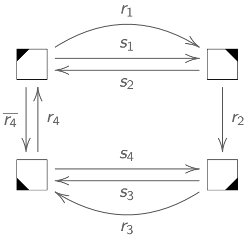
# Tietze transformations on our example




$$r_3 \circ r_2 \circ r_1 = \overline{r}_4 \qquad s_{j+1} \circ s_j = \mathrm{id} \qquad r_j \circ s_{j+1} \circ r_j = s_j$$
$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \qquad s_j \circ s_{j+1} = \mathrm{id} \qquad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

for $i \in \{1, \ldots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \ldots, 4\}$.

# Tietze transformations on our example



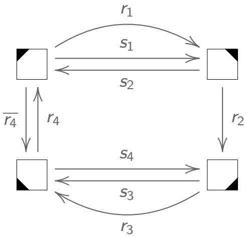

$$\overline{r}_4 \circ r_4 = \mathrm{id} \qquad s_{j+1} \circ s_j = \mathrm{id} \qquad r_j \circ s_{j+1} \circ r_j = s_j$$

$$r_4 \circ \overline{r}_4 = \mathrm{id} \qquad s_j \circ s_{j+1} = \mathrm{id} \qquad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \qquad r_3 \circ r_2 \circ r_1 = \overline{r}_4$$

for $i \in \{1, \dots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \dots, 4\}$.

# Tietze transformations on our example




$$\overline{r}_4 \circ r_4 = \mathrm{id} \quad s_{j+1} \circ s_j = \mathrm{id} \quad r_j \circ s_{j+1} \circ r_j = s_j$$

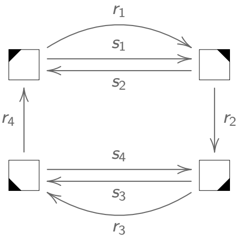
$$r_4 \circ \overline{r}_4 = \mathrm{id} \quad s_j \circ s_{j+1} = \mathrm{id} \quad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id}$$

for $i \in \{1, \ldots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \ldots, 4\}$.

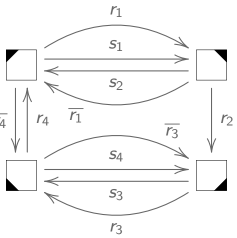
# Our example : the dihedral category $D_4^\bullet$

- We chose to add an inverse $\overline{r}_4$ for $r_4$ as the residual for $r_2$ after $s_2$
- $\overline{r}_4$ has to be an equational morphism
- The rewriting system with rules $\{r_2, r_4, \overline{r}_4\}$ is not terminating !
- However, it can be proven that the quotient (resp. localization) by $\{r_2, r_4\}$ and by $\{r_2, \overline{r}_4\}$ are isomorphic.



# Our example : the dihedral category $D_4^\bullet$

We change the presentation via Tietze transformations and we get :




$$s_{j+1} \circ s_j = \mathrm{id} \quad r_1 \circ s_2 \circ r_1 = s_1 \quad r_k \circ \overline{r}_k = \mathrm{id} \quad r_2 \circ r_1 = \overline{r}_3 \circ \overline{r}_4 \quad s_3 \circ r_2 = \overline{r}_4 \circ s_2$$
$$s_j \circ s_{j+1} = \mathrm{id} \quad \overline{r}_3 \circ s_3 \circ \overline{r}_3 = s_4 \quad \overline{r}_k \circ r_k = \mathrm{id} \quad r_3 \circ r_2 = \overline{r}_4 \circ \overline{r}_1 \quad r_2 \circ s_1 = s_4 \circ \overline{r}_4$$

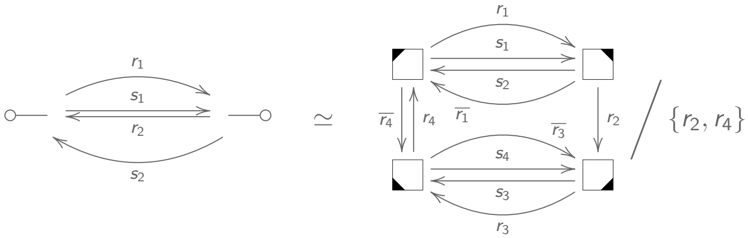
for $i \in \{1, \ldots, 4\}$, $j \in \{1, 3\}$ and $k \in \{1, 3, 4\}$

# Our example : the dihedral category $D_4^\bullet$

All assumptions are true.

## Theorem

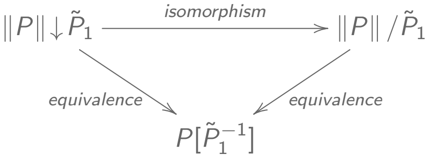
*The category $D_2^\bullet$ is isomorphic to the quotient $D_4^\bullet / \{r_2, r_4\}$, embeds fully and faithfully into the category $D_4^\bullet$, and is equivalent to the localization $D_4^\bullet[\{r_2, r_4\}^{-1}]$.*



# Conclusion

## Theorem

*If the presentation* $(P, \tilde{P}_1)$ *satisfies our properties, then*




- Florence Clerc, Samuel Mimram, *Presenting a Category Modulo a Rewriting System*, RTA 2015, Varsovie.
- Generalization on 2-categories to come