Embedding constructiveness into classical logic via polarization

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Motivation

Focusing and Double Negation

- Deal with contraction
- Deal with computing in classical logic using cut

Can the different literatures be related?

Outline

Focusing

Aim

Kuroda Translation

Further Work

Key idea behind focusing and polarization

- distinguish between invertible rules ($\Uparrow$: asynchronous phases) and non-invertible rules ($\Downarrow$: synchronous phases)
- Negative connectives are introduced during $\Uparrow$ phases and positive ones during $\Downarrow$ phases
- The multiplicative and additive versions of a connective have opposite polarity

$$\frac{\vdash B, \Gamma \quad \vdash C, \Gamma}{\vdash B \wedge^- C, \Gamma} \qquad \frac{\vdash B, \Gamma_1 \quad \vdash C, \Gamma_2}{\vdash B \wedge^+ C, \Gamma_1, \Gamma_2}$$

We next present the LKF and LJF focused proof systems of Liang & Miller (TCS 2009). These two proof systems describe focusing for classical and intuitionistic logics much as Andreoli (JLC 1992) did for linear logic.

LKF (cut-free proofs only)

We only considered formulas in NNF (negation normal form).

Decision, Reaction, Initial

$$\frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N}\ \textit{Release} \qquad \frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow .}\ \textit{Focus/Decide}$$

$$\frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C}\ \textit{Store} \qquad \frac{}{\vdash \neg P, \Theta \Downarrow P}\ \textit{Init}\ (P \text{ literal})$$

Asynchronous connectives

$$\frac{\vdash \Theta \Uparrow \Gamma, A \quad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}\ \wedge^- \qquad \frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B}\ \vee^-$$

Synchronous connectives

$$\frac{\vdash \Theta \Downarrow A \quad \vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \wedge^+ B}\ \wedge^+ \qquad \frac{\vdash \Theta \Downarrow A_i}{\vdash \Theta \Downarrow A_1 \vee^+ A_2}\ \vee^+$$

P positive, N negative, C positive formula or negative literal
Θ set of positive formulas or negative literals
Have omitted rules for $\exists$ (positive) and $\forall$ (negative)
End sequents: $\vdash . \Uparrow \Gamma$.

Important remarks on LKF

Theorem

Let B be a formula in LK and let $\hat{B}$ be any polarization of B. Then B is provable in LK if and only if $\hat{B}$ is provable in LKF.

Other Remarks

- Polarization influences the shape of proofs but not provability.
- Non-literal negative formulas are never weakened nor contracted: they are treated like formulas in linear logic.
- Positive formulas may be contracted using the decide rule
- First order quantifiers are easy to add ($\exists$ is positive and $\forall$ is negative).
- Multi-focusing is an easy extension to LKF.

LJF

Decision, Reaction and Initial Rules

$$\frac{N, \Gamma \Downarrow N \rightarrow R}{\Gamma, N; . \rightarrow R; .}\ F_l \qquad \frac{\Gamma \rightarrow\Downarrow P}{\Gamma; . \rightarrow P; .}\ F_r \qquad \frac{\Gamma; P \rightarrow R; .}{\Gamma \Downarrow P \rightarrow R}\ R_l \qquad \frac{\Gamma; . \rightarrow .; N}{\Gamma \rightarrow\Downarrow N}\ R_r$$

$$\frac{C, \Gamma; \Theta \rightarrow \mathcal{R}}{\Gamma; \Theta, C \rightarrow .; R}\ S_l \qquad \frac{\Gamma; \Theta \rightarrow D; .}{\Gamma; \Theta \rightarrow .; D}\ S_r$$

$$\frac{}{P, \Gamma \rightarrow\Downarrow P}\ I_r,\ \text{atomic } P \qquad \frac{}{\Gamma \Downarrow N \rightarrow N}\ I_l,\ \text{atomic } N$$

Introduction Rules

$$\frac{\Gamma \Downarrow A_i \rightarrow R}{\Gamma \Downarrow A_1 \wedge^- A_2 \rightarrow R}\ \wedge^- L \qquad \frac{\Gamma; \Theta \rightarrow .; A \quad \Gamma; \Theta \rightarrow .; B}{\Gamma; \Theta \rightarrow .; A \wedge^- B}\ \wedge^- R$$

$$\frac{\Gamma; \Theta, A, B \rightarrow \mathcal{R}}{\Gamma; \Theta, A \wedge^+ B \rightarrow R; .}\ \wedge^+ L \qquad \frac{\Gamma \rightarrow\Downarrow A \quad \Gamma \rightarrow\Downarrow B}{\Gamma \rightarrow\Downarrow A \wedge^+ B}\ \wedge^+ R$$

$$\frac{\Gamma; \Theta, A \rightarrow R; . \quad \Gamma; \Theta, B \rightarrow \mathcal{R}}{\Gamma; \Theta, A \vee B \rightarrow R; .}\ \vee L \qquad \frac{\Gamma \rightarrow\Downarrow A_i}{\Gamma \rightarrow\Downarrow A_1 \vee A_2}\ \vee R$$

with $\mathcal{R} = .; R$ or $R; .$

P positive, N negative

C negative formula or positive atom

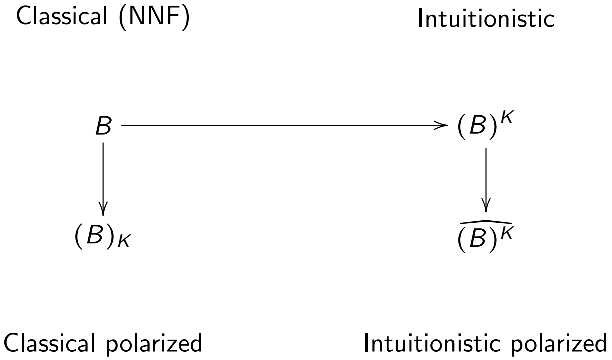
D positive formula or negative atom

Remarks on LJF

- LJT and LJQ are subparts of LJF: they are based on particular polarizations.
- Negative formulas on the left may be contracted using the decide rule (parallels observation in LKF).
- First order quantifiers are easy to add.
- Multi-focusing in LJF is more tricky to add.
- We define $\neg B$ as $B \supset \bot$ (hence, this is always a negative polarized formula).

Outline

Aim



Aim: find $\hat{.}$ and $(.)_K$ such that there is an isomorphism between:

- the proofs in LKF of $\vdash . \Uparrow (B)_K$
- the proofs in LKJ of $.;. \rightarrow \widehat{(B)^K}$

Outline

Arrows in the case of the Kuroda translation

- the Kuroda translation: A and B are first-order formulae $(B)^K = \neg\neg(B)'$ where

$$
\begin{array}{rr}
(P)' = P \text{ (P atom)} & (\neg P)' = \neg P \text{ (P atom)} \\
(A \wedge B)' = (A)' \wedge (B)' & (A \vee B)' = (A)' \vee (B)' \\
(\exists x.B)' = \exists x.(B)' & (\forall x.B)' = \forall x.\neg\neg(B)'
\end{array}
$$

- the translation from LK to LKF : $(B)_K = \partial^+(B)''$ where

$$
\begin{array}{rr}
(A \wedge B)'' = (A)'' \wedge^+ (B)'' & (A \vee B)'' = (A)'' \vee^+ (B)'' \\
(\exists x.B)'' = \exists x.(B)'' & (\forall x.B)'' = \forall x.\partial^+(B)'' \\
(P)'' = P \text{ (P positive atom)} & (\neg P)'' = \neg(P)''
\end{array}
$$

- the translation from LJ to LJF :

$$
\begin{array}{rr}
\widehat{A \wedge B} = \widehat{A} \wedge^+ \widehat{B} & \widehat{A \vee B} = \widehat{A} \vee^+ \widehat{B} \\
\widehat{\exists x.B} = \exists x.\widehat{B} & \widehat{\forall x.B} = \forall x.\widehat{B} \\
\widehat{P} = P \text{ (P positive atom)} & \widehat{\neg B} = \neg\widehat{B}
\end{array}
$$

Shape of the proof

Lemma

Given an LKF proof of $\vdash . \Uparrow (\phi)_K$*, any sequent of the form* $\vdash \Gamma(\Uparrow \text{ or } \Downarrow)\Delta$ *is such that* $\Gamma = \mathcal{U}, \mathcal{N}$ *where:*

- $\mathcal{N}$ *is a multiset of negated atoms*
- $\mathcal{U}$ *is a multiset of positive delayed formulae containing* $(\phi)_K$*.*

Proof of the lemma

The shape of a proof in LKF is

$$
\begin{array}{c}
\overline{\qquad\qquad\qquad} \; \mathit{init} \\
\vdots \\
\overline{\vdash \Gamma_1 \Uparrow \cdot} \; \mathit{Focus} \\
\vdots \\
\text{asynchronous phase}(\Uparrow) \\
\vdots \\
\dfrac{\text{synchronous phase}(\Downarrow)}{\vdash \Gamma_2 \Uparrow \cdot} \; \mathit{Focus} \\
\vdots \\
\overline{\vdash \Gamma_n \Uparrow \cdot} \; \mathit{Focus} \\
\vdots \\
\overline{\vdash \cdot \Uparrow \phi}
\end{array}
$$

Phase Correspondence

Theorem

B is provable in LK if and only if $(B)_K$ is provable in LKF if and only if $\widehat{(B)^K}$ is provable in LJF. Moreover there is a one-to-one correspondence between the bipoles of the proofs in LKF of $(B)_K$ and those in LJF of $\widehat{(B)^K}$.

Phase Correspondence: continued

Proof.

We give the phase correspondence: the proof is done by induction on the structure of the proof.

LJF	LKF
$\Gamma; \Delta \rightarrow .; R$	$\vdash \neg\overline{\Gamma} \Uparrow \neg\overline{\Delta}, \overline{R}$
$\Gamma; \Delta \rightarrow R; .$	$\vdash \neg\overline{\Gamma}, \overline{R} \Uparrow \neg\overline{\Delta}$
$\Gamma \Downarrow B \rightarrow R$	$\vdash \neg\overline{\Gamma}, \overline{R} \Downarrow \neg\overline{B}$
$\Gamma \rightarrow B \Downarrow$	$\vdash \neg\overline{\Gamma} \Downarrow \overline{B}$

and

LKF	LJF
$\vdash N_1, N_2, U_1, U_2 \Uparrow B, \Gamma$	$\begin{cases} \neg\underline{N_1}, \neg\underline{N_2}, \neg\underline{U_1}, \neg\underline{U_2}; \neg\underline{\Gamma} \rightarrow .; \underline{B} \\ \neg\underline{N_1}, \neg\underline{U_1}; \neg\underline{\Gamma}, \neg\underline{B} \rightarrow \underline{N_2}, \underline{U_2}; . \end{cases}$
$\vdash N_1, N_2, U_1, U_2 \Downarrow B$	$\begin{cases} \neg\underline{N_1}, \neg\underline{N_2}, \neg\underline{U_1}, \neg\underline{U_2} \rightarrow \underline{B} \Downarrow \\ \neg\underline{N_1}, \neg\underline{U_1} \Downarrow \neg\underline{B} \rightarrow \underline{N_2}, \underline{U_2} \end{cases}$

where (N_2, U_2) is at most one formula. □

Perspectives

Two major hypothesis:

- NNF (but yet not very restricting): easy to accommodate also general negation and implications.
- no cuts: what would happen if cuts were allowed?

Open question: what about cut elimination as a process?

Outline

Further Work

- Can all double-negation translations be described via polarization in LKF?
  - the Gödel-Gentzen translation : A and B are first-order formulae

$$
\begin{array}{rlrl}
(P)^{GG} & = \neg\neg P \text{ (P atom)} & (\neg P)^{GG} & = \neg P \text{ (P atom)} \\
(A \wedge B)^{GG} & = (A)^{GG} \wedge (B)^{GG} & (A \vee B)^{GG} & = \neg\neg\left((A)^{GG} \vee (B)^{GG}\right) \\
(\exists x.B)^{GG} & = \neg\neg\exists x.(B)^{GG} & (\forall x.B)^{GG} & = \forall x.(B)^{GG}
\end{array}
$$

  - the Krivine translation : A and B are first-order formulae $(B)^V = \neg(B)'$ where

$$
\begin{array}{rlrl}
(P)' & = \neg P \text{ (P atom)} & (P)' & = \neg\neg P \text{ (P atom)} \\
(A \wedge B)' & = (A)' \vee (B)' & (A \vee B)' & = (A)' \wedge (B)' \\
(\exists x.B)' & = \neg\exists x.\neg(B)' & (\forall x.B)' & = \exists x.(B)'
\end{array}
$$

Further Work

- Conversely, do all polarizations lead to double-negation translations?
- What about cut-elimination?
- Can we program with LK proofs without dealing with λ-terms?