Study of Double-Negation Translations and Focused Systems

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Ce stage s'inscrit dans le cadre de la théorie de la preuve. Les assistants de preuve ont connu un formidable essor et il en existe aujourd'hui un grand nombre tels que CoQ, Isabelle, HOL pour n'en citer que quelques-uns.

Le projet ProofCert s'appuie sur le constat simple que ces différents systèmes n'ont pas les mêmes formalismes et il vise donc à trouver des méthodes formelles pour pallier ce manque.

Ce stage fait partie intégrante du projet européen ProofCert. En effet, tout ce qui peut nous permettre de mieux comprendre les liens entre les différentes logiques est un apport, et c'est ce que vise ce stage puisque la logique est reconnue comme le langage universel des preuves.

Le point de vue que nous avons choisi de considérer pendant ce stage est plus particulièrement basé sur les aspects syntactiques de certaines logiques polarisées.

En effet, les logiques polarisées emploient un focus. Dans les preuves, le focus permet entre autre de garder une trace des formules qui ont été étudiées. Un pan de la théorie de la preuve s'appuie donc sur le focus pour pouvoir faire de la recherche de preuves plus efficace.

On veut montrer à terme que la logique polarisée est plus riche que la logique non polarisée car dans un système avec focus la logique classique peut avoir la même structure que la logique intuitionniste. Cela permettrait à terme de travailler dans la logique classique polarisée qui permet de prouver autant de choses que la logique classique, tout en bénéficiant des aspects constructifs de la logique intuitionniste.

Dans le cadre de ce travail, on s'intéresse tout particulièrement aux traductions basées sur la double-négation entre la logique classique et la logique intuitionniste. Ces traductions sont bien connues depuis les premiers travaux de Kolmogorov et permettent un premier support pour cette étude.

Nous avons étudié plusieurs traductions connues entre la logique classique et la logique intuitionniste, telles que Kuroda, Gödel-Gentzen. Ces traductions basées sur les doubles négations nous ont permis de mettre en évidence qu'en choisissant correctement les polarités d'une formule, les preuves obtenues en logiques polarisées classique et intuitionniste avaient la même structure.

Cela nous a amenés à étudier d'autres problématiques afin de conserver la cohérence entre les logiques classiques et intuitionnistes polarisées : nous avons ainsi repris les notations et avons cherché à mettre en place le focus sur plusieurs éléments à la fois.

Nous avons mis en lumière des similitudes que l'intuition nous permettait de suspecter dans le cadre syntactique : en effet, par construction de la logique polarisée, on distingue l'introduction réversible de celle non-réversible des connecteurs. On s'attend donc à ce qu'en choisissant correctement la syntaxe dans la logique classique polarisée, on soit en mesure de retrouver le contenu constructif de la logique intuitionniste.

Nous avons effectivement démontré que, dans le cas de certaines traductions par double négation, c'était possible.

En restant dans le même cadre, il est nécessaire de se pencher sur plus de traductions par double négation, mais également sur d'autres traductions telles que celle de Krivine.

De plus, ce problème mériterait d'autres ouvertures que celles permises par le seul cadre syntactique : sématique dénotationnelle avec LC, théorie des catégories etc. Ces approches pourraient éventuellement ouvrir une voie et méritent donc de s'y pencher un peu plus. Elles permettraient de dégager un schéma général au lieu d'avoir juste des cas particuliers et permettraient donc de mettre en lumière la puissance de LKF par rapport à LJ.

Ces travaux sont encore en cours puisque ce stage ne s'achève qu'à la mi-septembre.

Contents

Thanks

I would like to thank my supervisor, Dale Miller, for this opportunity to discover some aspects of proof theory and for the various conferences that contributed in making this experience a richer one.

I would also like to thank all the members of the team and all the PhD students for welcoming me. Among them, I shall thank especially Stéphane Lengrand for his useful advice and his time.

Finally I also thank all of those who gave me different perspectives on this internship and on PhD.

1 Introduction

ProofCert With the proof of the Four Color Theorem in 2005 by G.Gonthier and B.Werner (see [8]) in CoQ, proof assistants have proven to the mathematical community that they could be powerful and reliable tools. But given the success of proof theory and the various number of proof assistants (CoQ, Isabelle, HOL for instance), there is a need of common formalism. Indeed, there are different methods used and various formalisms for proofs. Proofcert (see [12]) is a project lead by Dale Miller that aims at finding such common ground.

Focused Proof Systems Focused proof systems seem very promising in terms of proof search. For instance, Andreoli studied focused proof systems for linear logic in [1] which provide a normal form for cut-free proofs in linear logic. Linear logic is well-suited for polarization as its connectives can be divided into those which have invertible rules and those which have non invertible rules.

Focusing also provides an environement for proof search in classical and intuitionistic logics (LK and LJ respectively). The key idea is to keep track of what has been already examined and what has not yet been looked through.

Problem Considered Focusing comes with polarization of formulas that distinguishes invertible and non invertible rules in the proof of that formula. Therefore, one could expect that it could also account for the constructive content of intuitionistic logic. Keeping that in mind, we have studied how focusing influenced proofs. We have relied on double-negation translations between LK and LJ that have been much studied in order to embed the constructive content of focused intuitionistic logic (LJF) into focused classical logic (LKF). These two logics were first introduced by C.Liang and D.Miller in [11].

We have succeeded in proving that it was possible to get same proof structures in LKF and in LJF, which is very encouraging in order to find an equivalent to lambda-calculus in LKF.

This has also lead us to considering some side-problems among which I will detail multifocusing in LJF.

Organization of this report Section 2 is devoted to detailing the background on LJF and LKF and some aspects of these polarized logics. Section 3 briefly explains the work that has been done. Finally in section 4 are a few comments on the work done and on its perspectives and ideas for future work.

2 Polarized Logics : LKF and LJF

2.1 General notions on focusing in LJF and in LKF

LKF and LJF were first introduced by C.Liang and D.Miller in [11].

The key idea behind focused logic is to distinguish the connectives depending on whether their introduction rules are reversible or not in sequent calculus. If the introduction rule is reversible, then the connective is considered to be negative. On the contrary, if the introduction rule is not reversible, then the connective is said to be positive. In LKF and in LJF, the formulae are of the same polarity as their top-level connectives which can be decided to be either positive or negative, regardless of the subformulae.

This allows to distinguish different phases during the proof. The negative connectives are treated during an asynchronous phase, whereas the positive connectives are treated during a

Decision, Reaction, Initial

$$\frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N}\ Release \qquad \frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow .}\ Focus/Decide$$

$$\frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C}\ Store \qquad \frac{}{\vdash \neg P, \Theta \Downarrow P}\ Init \text{ (literal } P\text{)}$$

Asynchronous Connectives

$$\frac{}{\vdash \Theta \Uparrow \Gamma, \neg \mathcal{F}}\ absurd \qquad \frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, \neg \mathcal{T}}\ trivial \qquad \frac{\vdash \Theta \Uparrow \Gamma, A \quad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}\ \wedge^-$$

$$\frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B}\ \vee^- \qquad \frac{\vdash \Theta \Uparrow \Gamma, B, \neg A}{\vdash \Theta \Uparrow \Gamma, A \supset^- B}\ \supset^- \qquad \frac{\vdash \Theta \Uparrow \Gamma, A}{\vdash \Theta \Uparrow \Gamma, \forall x A}\ \forall$$

Synchronous Connectives

$$\frac{}{\vdash \Theta \Downarrow \mathcal{T}}\ \mathcal{T} \qquad \frac{\vdash \Theta \Downarrow A \quad \vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \wedge^+ B}\ \wedge^+ \qquad \frac{\vdash \Theta \Downarrow A_i}{\vdash \Theta \Downarrow A_1 \vee^+ A_2}\ \vee^+$$

$$\frac{\vdash \Theta \Downarrow A[t/x]}{\vdash \Theta \Downarrow \exists x A}\ \exists \qquad \frac{\vdash \Theta \Downarrow \neg A}{\vdash \Theta \Downarrow A \supset^+ B}\ \supset^+ \qquad \frac{\vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \supset^+ B}\ \supset^+$$

Figure 1: The Classical Sequent Calculus LKF. Here, P is positive, N is negative, C is a positive formula or a negative literal, Θ consists of positive formulae and negative literals, and x is not free in Θ, Γ. Endsequents have the form $\vdash . \Uparrow \Gamma$.

synchronous phase. Therefore, if choices have to be made, they are made during the synchronous phase.

The focus allows to keep track of the formulae that have been studied and examined. The focus exists only during the synchronous phases. The proof then alternates with an asynchronous phase and so on.

The work I have done has been concentrated on single-focused systems : LKF and LJF. That is to say that I have only allowed to work on a single formula during synchronous phase. There also exist multi-focused systems where the focus may contain more than one formula. This is possible in the classical setting and some work has also been done on them.

2.2 LKF

The notations for LKF have been changed in order to be more easily readable. The calculus using these notations is presented in figure 1.

Soundness and completeness with respect to LK The following result has been proven in [11]. I have decided here to introduce first LKF and then LJF as the rules for LJF can be seen as the rules for LKF where the right-hand side of the sequent is reduced to zero or one formula. But it was done the other way in [11] where the following result is shown by embedding classical logic into LJF where a similar theorem had previously been proven.

Theorem 1. *LKF is sound and complete with respect to classical logic.*

This result is very strong as it means that given any formula ϕ provable in LK, given any polarization of ϕ noted $\widehat{\phi}$, then $\widehat{\phi}$ is provable in LKF. To go from ϕ to $\widehat{\phi}$, one has to decide for all the connectives and all the atoms whether they are positive or negative. It should be noted that if the same connective is used twice in ϕ, they still can have different polarities in $\widehat{\phi}$.

The converse is also true : if $\widehat{\phi}$ is provable in LKF, then when forgeting the polarities to get ϕ, ϕ is provable in LK.

Therefore, given two polarized formulae in LKF ϕ_1 and ϕ_2 such that they only differ in terms of polarities, ϕ_1 is provable in LKF if and only if ϕ_2 is provable in LKF.

Shape of a proof in LKF A proof in LKF can be decomposed into several phases :

- first phase : from an init to the first rule of the form $\vdash \Gamma_1 \Uparrow .$
- second phase : $\forall i = 1..n-1$ from a rule of the form $\vdash \Gamma_i \Uparrow .$ to the next rule of the form $\vdash \Gamma_{i+1} \Uparrow .$
  The subproof from one rule of the form $\vdash \Gamma_i \Uparrow .$ to the next rule of the form $\vdash \Gamma_{i+1} \Uparrow .$ is named a bipole after the definition given by JM Andreoli in [2] and it is the succession of a synchronous phase beginning by a focus rule and an asynchronous phase beginning by a release rule.
- third phase : from the last rule of the form $\vdash \Gamma_n \Uparrow .$ to $\vdash . \Uparrow \partial^+(\phi)_K$

Delays in LKF Delays are means of forcing the polarity of a formula. There are two different delays : a positive one (∂^+) and a negative one (∂^-). These allow to change phases and thus focusing. Please note that for any LKF formula B, $\partial^- B$, $\partial^+ B$ and B are logically equivalent, the only difference is that $\partial^- B$ is negative, $\partial^+ B$ is positive and we do not know the polarity of B a priori.

As $\partial^- B = true \wedge^- B$ and $\partial^+ B = true \wedge^+ B$, the rules for ∂^+ and for ∂^- are :

$$\frac{\vdash \theta \Uparrow \Gamma, B}{\vdash \theta \Uparrow \Gamma, \partial^- B} \qquad \frac{\vdash \theta \Downarrow \Gamma, B}{\vdash \theta \Downarrow \Gamma, \partial^+ B}$$

2.3 LJF

When considering the rules of LKF, one notes that the introduction rules for $\vee^-$ and thus for $\supset^-$ requires to have two formulae under focus. In intuitionistic logic, there can be at most one formula in the right-hand side of the sequent, which forces to adapt the connectives and the rules that are used. In particular, the sequents in LJF have a left-hand side and a right-hand side, whereas the sequents in LKF were single-sided.

This also means that a choice has to be made for the connectives : $\vee^-$ is no longer kept, and only its constructive version $\vee^+$ exists in LJF. On the contrary, $\supset^-$ can be kept given that in LKF $A \supset^- B \equiv (\neg A) \vee^- B$ which allows to take the common sequent rule for $\supset$.

As previously mentionned, the notations for the LKF calculus were changed in order to be more intuitive and readable. To maintain coherence with the LKF calculus, I suggested to change the notations for LJF, which D.Miller accepted. The system of rules for LJF using the new notation is presented in figure 2.

The following result has been proven in [11]

Theorem 2. *LJF is sound and complete with respect to intuitionistic logic.*

Decision and Reaction Rules

$$\frac{N, \Gamma \Downarrow N \to R}{\Gamma, N; . \to R; .}\ F_l \qquad \frac{\Gamma \to\Downarrow P}{\Gamma; . \to P; .}\ F_r \qquad \frac{\Gamma; P \to R; .}{\Gamma \Downarrow P \to R}\ R_l \qquad \frac{\Gamma; . \to .; N}{\Gamma \to\Downarrow N}\ R_r$$

$$\frac{C, \Gamma; \Theta \to .; R}{\Gamma; \Theta, C \to .; R}\ S_l \qquad \frac{C, \Gamma; \Theta \to R; .}{\Gamma; \Theta, C \to R; .}\ S_l \qquad \frac{\Gamma; \Theta \to D; .}{\Gamma; \Theta \to .; D}\ S_r$$

Initial Rules

$$\frac{}{P, \Gamma \to\Downarrow P}\ I_r,\ \text{atomic } P \qquad \frac{}{\Gamma \Downarrow N \to N}\ I_l,\ \text{atomic } N$$

Introduction Rules

$$\frac{}{\Gamma; \Theta, false \to .; R}\ falseL \qquad \frac{}{\Gamma; \Theta, false \to R; .}\ falseL$$

$$\frac{\Gamma; \Theta \to .; R}{\Gamma; \Theta, true \to .; R}\ trueL \qquad \frac{\Gamma; \Theta \to R; .}{\Gamma; \Theta, true \to R; .}\ trueL \qquad \frac{}{\Gamma \to\Downarrow true}\ trueR$$

$$\frac{\Gamma \Downarrow A_i \to R}{\Gamma \Downarrow A_1 \wedge^- A_2 \to R}\ \wedge^- L \qquad \frac{\Gamma; \Theta \to .; A \quad \Gamma; \Theta \to .; B}{\Gamma; \Theta \to .; A \wedge^- B}\ \wedge^- R$$

$$\frac{\Gamma; \Theta, A, B \to .; R}{\Gamma; \Theta, A \ \wedge^+ B \to .; R}\ \wedge^+ L \qquad \frac{\Gamma; \Theta, A, B \to R; .}{\Gamma; \Theta, A \ \wedge^+ B \to R; .}\ \wedge^+ L \qquad \frac{\Gamma \to\Downarrow A \quad \Gamma \to\Downarrow B}{\Gamma \to\Downarrow A \wedge^+ B}\ \wedge^+ R$$

$$\frac{\Gamma; \Theta, A \to .; R \quad \Gamma; \Theta, B \to .; R}{\Gamma; \Theta, A \vee B \to .; R}\ \vee L \qquad \frac{\Gamma; \Theta, A \to R; . \quad \Gamma; \Theta, B \to R; .}{\Gamma; \Theta, A \vee B \to R; .}\ \vee L \qquad \frac{\Gamma \to\Downarrow A_i}{\Gamma \to\Downarrow A_1 \vee A_2}\ \vee R$$

$$\frac{\Gamma \to\Downarrow A \quad \Gamma \Downarrow B \to R}{\Gamma \Downarrow A \supset B \to R}\ \supset L \qquad \frac{\Gamma; \Theta, A \to .; B}{\Gamma; \Theta \to .; A \supset B}\ \supset R$$

$$\frac{\Gamma; \Theta, A \to .; R}{\Gamma; \Theta, \exists y A \to .; R}\ \exists L \qquad \frac{\Gamma; \Theta, A \to R; .}{\Gamma; \Theta, \exists y A \to R; .}\ \exists L \qquad \frac{\Gamma \to\Downarrow A[t/x]}{\Gamma \to\Downarrow \exists x A}\ \exists R$$

$$\frac{\Gamma \Downarrow A[t/x] \to R}{\Gamma \Downarrow \forall x A \to R}\ \forall L \qquad \frac{\Gamma; \Theta \to .; A}{\Gamma; \Theta \to .; \forall y A}\ \forall R$$

Figure 2: The Intuitionistic Sequent Calculus LJF. Here, P is positive, N is negative, C is a negative formula or positive atom, and D a positive formula or negative atom. Other formulae are arbitrary. Also, y is not free in Γ, Θ, or R.

2.4 Other polarized logics

2.4.1 LKT and LKQ

LKF and LJF were introduced as a way of unifying LKT and LKQ developed in [3]. T stands for head ("tête") whereas Q stands for queue. LKT and LKQ are focused systems in which proofs correspond respectively to call-by-name and call-by-value evaluations of programs (see [15] for further details). LKF is able to account for both.

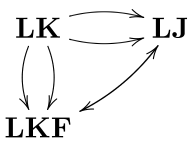
There are other ways of polarizing formulae. I should mention two of them.

2.4.2 Using a shift

Some systems only allow positive connectives between positive subformulae. Therefore, in order to be able to deal with both reversible and non-reversible rules, a shift ($\updownarrow$) has been introduced by girard in [6]. Thanks to this shift, we get the following syntax :

$$\begin{array}{rcl} P, P_1, P_2 & ::= & P|P_1 \vee^+ P_2|P_1 \wedge^+ P_2|\exists x.P|\updownarrow N \\ N, N_1, N_2 & ::= & N|N_1 \vee^- N_2|N_1 \wedge^- N_2|\forall x.N|\updownarrow P \end{array}$$

Together with Stéphane Lengrand, we discussed how for each translation from LK to LJ, a polarization from LK to shifted classical logic could be found that would allow to keep similar proof structures.



We have found a translation $\overline{\cdot}$ from shifted classical logic to LJ that seemed to allow to have close proofs for B in shifted classical logic and $\overline{B}$ in LJ.

As there is no shift in LKF and LJF, this translation should be adapted. We did that by designing two translations depending on whether a positive or a negative translation was expected : $\overline{\cdot}^+$ and $\overline{\cdot}^-$.

2.4.3 In LC

In constructive logic (LC) designed by Girard in [5], the polarity of a formula depends on the entire subformulae. For instance $A \vee B$ is positive if both A and B are positive and negative otherwise. The polarities of the formulae depending on all cases can be found in [5]. The polarities are similar to the truth tables (where a positive polarity stands for false) as the tautologies (for instance the de morgan dualities) were kept as isomorphims in the semantics.

The polarity is not on the connective but on the entire formula, whereas in LKF and LJF, the connectives exist in both versions : a positive one and a negative one, independantly of their subformulae.

The idea that lead to LC is a very rich one that will be further discussed in section 4.2.1.

2.5 Multifocusing in LJF

In both LKF and LJF, only one formula is allowed under the focus during the synchronous phase. This is called single-focusing.

It is possible though that focusing on formula A and then on formula B gives the same result as focusing first on B and then on A. In this setting, it is possible to focus on both formulae at the same time. This is called multifocusing and it corresponds to parallel execution.

Multifocusing exists in LKF and therefore it was expected to also exist in LJF.

The single-focused rule for implication in LJF is the following :

$$\frac{\Gamma \rightarrow\Downarrow A \quad \Gamma \Downarrow B \rightarrow R}{\Gamma \Downarrow A \supset B \rightarrow R} \supset L$$

Assuming the last sequent is multifocused, one would get $\Gamma \Downarrow \Delta, A \supset B \rightarrow R$ where Δ would be duplicated or split between the two upper sequents. One would then get $\Gamma \Downarrow \Delta_1 \rightarrow\Downarrow A$ and $\Gamma \Downarrow B, \Delta_2 \rightarrow R$ where Δ_1 and Δ_2 depend on whether Δ is split or duplicated. We would therefore have expected the multifocusing to be on both sides of the sequents.

I have tried several possibilities, but I kept on having troubles with the Decision and Reaction rules. How could I make sure that the two sides of the sequents were ready at the same time for a change of phases ? There was one possibility : if $\Delta_1 = \emptyset$ and $\Delta_2 = \Delta$, we would need multifocusing only on one side.

Whatever we tried, we could not find any satisfying way to deal with the change of phases. The intuition tells that there is a way of multifocusing in intuitionistic logic, but maybe that our intuition is wrong. Therefore the reason why we could not find any way of multifocusing in LJF remains an open question.

3 Some double-negation translations and their embedding into LKF/LJF

All the formulae in classical logic (focused or unfocused) that are considered are in negative normal form (NNF).

3.1 Double-negation translations

Many first-order formulae that are provable in classical logic are no longer true in intuitionistic logic. This is the case for various famous examples such as the principle of excluded middle : $A \vee \neg A$ is provable in classical logic but not in intuitionistic logic.

There exist some translations $\widehat{\ }$ such that for any formula ϕ and for any set of formulae Γ :

$$\Gamma \vdash_K \phi \Leftrightarrow \widehat{\Gamma} \vdash_I \widehat{\phi}$$

where $\widehat{\phi_1, ..., \phi_n} = \widehat{\phi_1}, ..., \widehat{\phi_n}$

Many such translations exist and have been developed since Kolomogorov studied the connection between classical and intuitionistic logics in 1925 [9]. We have studied two of them and we are trying to work on a third one (see [4] for more examples of them).

- the Kuroda translation : A and B are first-order formulae $(B)^K = \neg\neg(B)'$ where

$$(P)' = P \text{ (P atom)} \quad (\neg P)' = \neg P \text{ (P atom)}$$
$$(A \wedge B)' = (A)' \wedge (B)' \quad (A \vee B)' = (A)' \vee (B)'$$
$$(\exists x.B)' = \exists x.(B)' \quad (\forall x.B)' = \forall x.\neg\neg(B)'$$

- the Gödel-Gentzen translation : A and B are first-order formulae

$$(P)^{GG} = \neg\neg P \text{ (P atom)} \qquad (\neg P)^{GG} = \neg P \text{ (P atom)}$$
$$(A \wedge B)^{GG} = (A)^{GG} \wedge (B)^{GG} \quad (A \vee B)^{GG} = \neg\neg\left((A)^{GG} \vee (B)^{GG}\right)$$
$$(\exists x.B)^{GG} = \neg\neg\exists x.(B)^{GG} \qquad (\forall x.B)^{GG} = \forall x.(B)^{GG}$$

- the Krivine translation : A and B are first-order formulae $(B)^K = \neg(B)'$ where

$$\begin{array}{cc} (P)' = \neg P \text{ (P atom)} & (P)' = \neg\neg P \text{ (P atom)} \\ (A \wedge B)' = (A)' \vee (B)' & (A \vee B)' = (A)' \wedge (B)' \\ (\exists x.B)' = \neg\exists x.\neg(B)' & (\forall x.B)' = \exists x.(B)' \end{array}$$

The Curry-Howard isomorphism gives a slighty different way of viewing double-negation translation from the historical approach. Indeed, double-negation translations are equivalent to continuation-passing style (CPS) translations (see [14] for details).

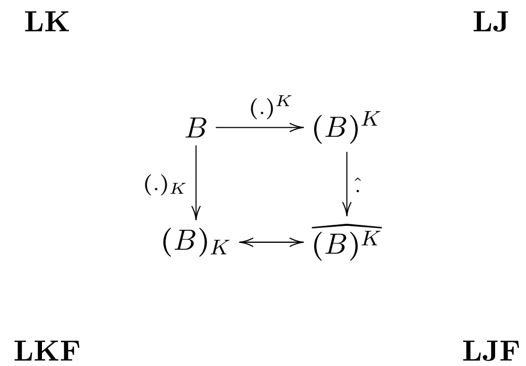
Let us briefly recall what CPS is (more details can be found in [10]). In lambda-calculus, one may want to reduce expressions to some normal form using β-reduction. Unfortunately, there is no such normal form if using standard β-reduction. But one may decide to use only call-by-name (the biggest expression that can be reduced is reduced) or call-by-value (where arguments are values) reductions. CPS is a way of unifying these two reductions.

3.2 Closer study of the Kuroda translation

3.2.1 Aim and definitions

The aim of this internship was to study how the proofs behave and relate in four different logics : LK, LJ, LKF and LJF. I have first studied the link between LKF and LJ using the already-studied double negation translations between LK and LJ. Although this allows to understand the way the different systems and the double negations translations behave, what we were trying to obtain was a result on focused systems.

In the case of the Kuroda translation, I have designed a polarization for the formulae in LK and a polarization for the formulae in LJ such that the proofs obtained in LKF and LJF have the same structure. It can be represented via the diagram :



Where I define the arrows to be :

- the Kuroda translation from a formula in LK to a formula in LJ given in subsection 3.1
- the translation from LK to LKF : $(B)_K = \partial^+(B)''$ where

$$\begin{array}{cc} (A \wedge B)'' = (A)'' \wedge^+ (B)'' & (A \vee B)'' = (A)'' \vee^+ (B)'' \\ (\exists x.B)'' = \exists x.(B)'' & (\forall x.B)'' = \forall x.\partial^+(B)'' \\ (P)'' = P \text{ (P positive atom)} & (\neg P)'' = \neg(P)'' \end{array}$$

- the translation from LJ to LJF :

$$\widehat{A \wedge B} = \widehat{A} \wedge^+ \widehat{B} \quad \widehat{A \vee B} = \widehat{A} \vee^+ \widehat{B}$$
$$\widehat{\exists x.B} = \exists x.\widehat{B} \quad \widehat{\forall x.B} = \forall x.\widehat{B}$$
$$\widehat{P} = P \text{ (P positive atom)} \qquad \widehat{\neg B} = \neg \widehat{B}$$

Let us now give an example to illustrate our work. These are the proofs of the excluded-middle after the given translations. The synchronous phases have been coloured in blue in order to ease the reading. One notes very quickly that the bipoles look similar in the LJF-proof and in the LKF-proof.

in LJF :

$$\begin{array}{c}
\hline
\neg(A \vee^+ \neg A), A \rightarrow\Downarrow A \\
\hline
\neg(A \vee^+ \neg A), A \rightarrow\Downarrow A \vee^+ \neg A \\
\hline
\neg(A \vee^+ \neg A), A \Downarrow \neg(A \vee^+ \neg A) \rightarrow . \\
\hline
\neg(A \vee^+ \neg A), A; . \rightarrow .; . \\
\hline
\neg(A \vee^+ \neg A), A; . \rightarrow .; . \\
\hline
\neg(A \vee^+ \neg A); A \rightarrow .; . \\
\hline
\neg(A \vee^+ \neg A); . \rightarrow \neg A; . \\
\hline
\neg(A \vee^+ \neg A); . \rightarrow .; \neg A \\
\hline
\neg(A \vee^+ \neg A) \rightarrow\Downarrow \neg A \\
\hline
\neg(A \vee^+ \neg A) \rightarrow\Downarrow A \vee^+ \neg A \\
\hline
\neg(A \vee^+ \neg A) \Downarrow \neg(A \vee^+ \neg A) \rightarrow . \\
\hline
\neg(A \vee^+ \neg A); . \rightarrow .; . \\
\hline
.; \neg(A \vee^+ \neg A) \rightarrow .; . \\
\hline
.; . \rightarrow .; \neg\neg(A \vee^+ \neg A)
\end{array}$$

in LKF :

$$\begin{array}{c}
\hline
\vdash \partial^+(A \vee^+ \neg A), \neg A \Downarrow A \\
\hline
\vdash \partial^+(A \vee^+ \neg A), \neg A \Downarrow A \vee^+ \neg A \\
\hline
\vdash \partial^+(A \vee^+ \neg A), \neg A \Downarrow \partial^+(A \vee^+ \neg A) \\
\hline
\vdash \partial^+(A \vee^+ \neg A), \neg A \Uparrow . \\
\hline
\vdash \partial^+(A \vee^+ \neg A) \Uparrow \neg A \\
\hline
\vdash \partial^+(A \vee^+ \neg A) \Downarrow \neg A \\
\hline
\vdash \partial^+(A \vee^+ \neg A) \Downarrow A \vee^+ \neg A \\
\hline
\vdash \partial^+(A \vee^+ \neg A) \Downarrow \partial^+(A \vee^+ \neg A) \\
\hline
\vdash \partial^+(A \vee^+ \neg A) \Uparrow . \\
\hline
\vdash . \Uparrow \partial^+(A \vee^+ \neg A)
\end{array}$$

3.2.2 Mathematical results

I shall now give the main result, the key lemma and an idea of the proofs. A detailed version of the proofs can be sent upon request.

Lemma 1. *Given an LKF proof of* $\vdash . \Uparrow (\phi)_K$*, any sequent of the form* $\vdash \Gamma(\Uparrow \text{ or } \Downarrow)\Delta$ *is such that* $\Gamma = \mathcal{U}, \mathcal{N}$ *where :*

- $\mathcal{N}$ *is a multiset of negated atoms*
- $\mathcal{U}$ *is a multiset of positive delayed formulae containing* $(\phi)_K$.

Proof. To begin with, this is the case for all sequents of the form $\vdash \Gamma \Uparrow .$

Indeed, there cannot be any negative formulae which are not negated atoms given the restrictions on the use of the store rule. And if there was any positive formula B that was not delayed, and as we are in an asynchronous phase, we would not be able to use the release rule once the store rule has been applied on B. Therefore we would introduce some negative connectives which is not allowed as $(\phi)_K$ contains only positive connectives.

We are then able to conclude on this lemma given the fact that Γ can only decrease throughout the proof. The shape of the proof allows us to apply the previously shown result. □

Theorem 3. *B is provable in LK if and only if $(B)_K$ is provable in LKF if and only if $\widehat{(B)^K}$. Moreover there is a one-to-one correspondance between the bipoles of the abstracted proofs in LKF of $(B)_K$ and those in LJF of $\widehat{(B)^K}$.*

Proof. This is a very long and tedious proof as it relies on disjunction of cases on all rules that can be applied when establishing the correspondance from LKF to LJF and then from LJF to LKF. Thanks to lemma 1 we are able to reduce the number of rules that can be applied.

I should only give the correspondance between the sequents :

LJF	LKF
$\Gamma; \Delta \rightarrow .; R$	$\vdash \neg\overline{\Gamma} \Uparrow \neg\overline{\Delta}, \overline{R}$
$\Gamma; \Delta \rightarrow R; .$	$\vdash \neg\overline{\Gamma}, \overline{R} \Uparrow \neg\overline{\Delta}$
$\Gamma \Downarrow B \rightarrow R$	$\vdash \neg\overline{\Gamma}, \overline{R} \Downarrow \neg\overline{B}$
$\Gamma \rightarrow B \Downarrow$	$\vdash \neg\overline{\Gamma} \Downarrow \overline{B}$

and

LKF	LJF
$\vdash N_1, N_2, U_1, U_2 \Uparrow B, \Gamma$	$\begin{cases} \neg\underline{N_1}, \neg\underline{N_2}, \neg\underline{U_1}, \neg\underline{U_2}; \neg\underline{\Gamma} \rightarrow .; \underline{B} \\ \neg\underline{N_1}, \neg\underline{U_1}; \neg\underline{\Gamma}, \neg\underline{B} \rightarrow \underline{N_2}, \underline{U_2}; . \end{cases}$
$\vdash N_1, N_2, U_1, U_2 \Downarrow B$	$\begin{cases} \neg\underline{N_1}, \neg\underline{N_2}, \neg\underline{U_1}, \neg\underline{U_2} \rightarrow \underline{B} \Downarrow \\ \neg\underline{N_1}, \neg\underline{U_1} \Downarrow \neg\underline{B} \rightarrow \underline{N_2}, \underline{U_2} \end{cases}$

where (N_2, U_2) is at most one formula.

□

3.2.3 Intuition

I should now give an intuition of why this translation was expected to give similar structure of the proofs. Let us first study unfocused systems. The key difference is that LJ only allows to have at most one formula on the right handside of the sequent. In LK formulae can be copied as often as needed, but this is not possible in LJ. In order to copy a formula on the right, one needs to first put it on the left handside where it can be copied and then put it back on the right handside. This is done thanks to double negations.

In order to force similar proof structure, we have to keep this which is why we use positive delays : they allow to put formulae in the unfocused part where they can be copied through the focus rule.

3.3 Closer study of the Gödel-Gentzen translation

Remember that the intuition is to relie on positive delays to force a formula to go to the unfocused part where in LJ it would be forced by a double-negation to go to the left-hand side of a sequent.

Therefore, assuming that the translation from LK to LJ adds a double negation in front of a connective $\star$, we would put a positive delay in LKF. The problem is the polarity of $\star$ in LKF. Should it remain positive ? In this case we could get rid of it in the asynchronous phase and therefore we would not have to put it in the unfocused part. So we would rather have the negative version of $\star$ in both LKF and LJF : having a different polarity forces to change phases which would correspond in LJ to moving a formula from the right-hand side to the left-hand side. This is unfortunately impossible for $\vee$ as there is no $\vee^-$ in LJF.

There are many possibilities. I am currently working on these in order to find out if any of them are working. We expect that at least one of these translations should work.

One possibility is :

$$\begin{aligned} (N)_{GG} &= \partial^+ N \text{ (N negative atom)} & (\neg N)_{GG} &= \neg N \text{ (N negative atom)} \\ (A \wedge B)_{GG} &= (A)_{GG} \wedge^- (B)_{GG} & (A \vee B)_{GG} &= \partial^-((A)_{GG} \vee^- (B)_{GG}) \\ (\exists x.B)_{GG} &= \exists x.\partial^-((B)_{GG}) & (\forall x.B)_{GG} &= \forall x.(B)_{GG} \end{aligned}$$

The problem here is that we do not know how to deal with a negative version of $\vee$ in LJF.

An other possibility that I suggest is :

$$(N)_{GG} = \partial^- N \text{ (N positive atom)} \quad (\neg N)_{GG} = \neg N \text{ (N positive atom)}$$
$$(A \wedge B)_{GG} = (A)_{GG} \wedge^+ (B)_{GG} \quad (A \vee B)_{GG} = \partial^-((A)_{GG} \vee^+ (B)_{GG})$$
$$(\exists x.B)_{GG} = \exists x.\partial^-((B)_{GG}) \quad (\forall x.B)_{GG} = \forall x.(B)_{GG}$$

By forcing the change of phases, we emulate double negations. But do we really need that ? By trying these on several examples, we expect to be able to come up with an appropriate solution. There are probably some parts of these translations that do work and others that do not work but hopefully, by using them, we can find one translation that works.

In that setting, the Kolmogorov translation is extremely interesting as it looks very close to the Gödel-Gentzen translation. Therefore, it should work in a similar way so it gives us the means to check that this intuition is reliable.

4 Comments and future developments

4.1 Open questions

4.1.1 Multifocusing in LJF

What we have begun to show is extremely strong : for all formulae B provable in LK, there is a one-to-one correspondance between the phases of the proof of $\widehat{(B)^K}$ in LJF and of the proof of $(B)_K$ in LKF. Therefore, we would expect to have the same possibilities in term of multifocusing.

As explained in subsection 2.5, we did not manage to find a way to make multifocusing work in LJF. The reason of this is still an open question. Maybe that this is due to the fact that we have only established that LJF and a subsection of LKF can behave in similar ways. This could be further studied through some semantic analysis.

There are other arguments supporting the existence of multifocusing in LJF. Multifocusing corresponds concurrent search for proof (see [13]). I could not find any counter-examples but we only have an intuition that multifocusing should also exist in LJF. If it does not work at all, it could indicate an important difference between the two logics that would force us to understand them differently.

4.1.2 Other types of translations between LK and LJ

The intuition of why double-negation translations work fine is explained in subsubsection 3.2.3. It has to do with duplicating formulas which is possible only on the left hand-side of the sequent in the intuitionistic setting, whereas it is always possible in the classical one. Thanks to double negations, it is possible in LJ to duplicate a formula on the left-hand side of the sequent and then use it in the right-hand side of the sequent.

This should be harder with single-negation translations such as the Krivine one (see [4]). The intuition given does not apply anymore, but there can be some possiblities using the dualities $\vee^+/\wedge^-$ and $\vee^-/\wedge^+$ or by designing an LKF-calculus using two-sided sequents.

Designing a two-sided LKF sequent could also help us understand why we could not find a way of multifocusing.

4.2 Related Work

In this work, we have only considered the syntactic point of view but others could be considered such as semantic aspects.

4.2.1 Constructive logic (LC) from Girard

In [5], Girard introduces a polarized logic LC that was already mentionned in subsubsection 2.4.3 . What is particularly interesting is that Girard's idea came first through the semantics and that he then translated it into the syntax. This allows to have some nice mathematical structure. Let us detail this.

It first appeared to him that there was no proof theory in LK. For instance, there is no cut-elimination procedure in LK. Indeed, the cuts are non commutative in LK as shown by Lafont's counterexample in [7]. He looked for a polarized logic that would allow to get rid of this non determinism.

He designed a semantic based on correlation spaces : proofs are cliques in such correlation spaces.

Definition 1. *A* coherence space *S is a couple* $(|S|, \mathcal{R}_S)$ *where:*

- $|S|$ *is a set of objects*
- $\mathcal{R}_S$ *is a reflexive symmetric relation* $\mathcal{R}_S \subset |S| \times |S|$

A clique *u in a coherence space S is a subset of $|S|$ such that*

$$\forall x, y \in u \quad x \mathcal{R}_S y$$

Definition 2. *A* negative correlation space *is a coherence space S together with :*

- *a clique $\perp$ of S satisfying neutrality*
- *a clique $+$ of $(S \parr S) \multimap S$ satisfying commutativity and associativity*

A positive correlation space *is the linear negation of a negative correlation space. For a detailed definition, please refer to [5].*

By sticking to this structure as a logic, he got the syntax of LC. LC has some interesting similarities with intuitionistic logic thanks to focusing, which enables Girard to give a translation from LC to LJ in [5] in terms of semantics.

4.2.2 Categorical aspects

The aim of this work is to establish a very strong link between LKF and LJF, which could give us a hint of a calculus in LKF similar to lambda-calculus in LJK.

There is of course a lot of work that needs to be done and the study of double-negation translations is only the beginning of it. Moreover, what we are trying to show is stronger than what was done in the case of unfocused logics. Indeed, the result in focused logics is not only on provability but also on the structures of the proofs.

Unfortunately, it is very hard to find the correct translations for each double-negation translations, as shown with the case of the Gödel-Gentzen translation in subsection 3.3. We have only studied the syntactic aspects, but the categorical, denotational or semantical ones could help us better understand what exactly makes this work and therefore what translations are suitable between LK and LKF and LJ and LJF in order to keep similar structures. Maybe that finding these underlying structures could even help us succeed in greater goals such as finding a calculus in LKF.

What is puzzling is that LK and LJ have very different structures which means thanks to [11] that LKF and LJF have different structures, but how different exactly ?

5 Conclusion

We have shown that focusing could account for the constructive content of intuitionistic logic, even in the classical setting. Indeed, by studying some double-negation translations between LK and LJ, we have shown that it was possible to find polarizations from LK to LKF and from LJ to LJF that allowed to get the same proof structures in LJF and in LKF.

This is a strong indication of the possibilities offered by focused systems in terms of proof search. Hopefully this could lead to a successfull and efficient calculus in LKF, similar to lambda-calculus in LJ.

However, there are many other aspects to study. For instance this result is very encouraging but what about the structure of proofs in LKF ?

There are also many questions regarding the syntactic aspects. One of them is cut elimination which we have not studied here. Cut elimination is not a deterministic procedure in LK so the way cuts behave in LKF is an important point that could eventually indicate that we are heading towards the wrong direction.

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