Presentations Modulo

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Introduction

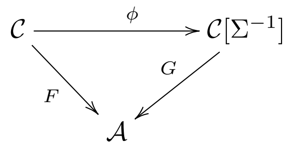
- presentations of monoids = string rewriting systems
- we first investigate a small generalization to presentations of categories
-

1 Localization and quotient of categories, category of fractions

1.1 Localization

Definition 1. Let $\mathcal{C}$ be a category. Let Σ be a set of morphisms of $\mathcal{C}$.

A *localization* of $\mathcal{C}$ by Σ is given by a category $\mathcal{C}[\Sigma^{-1}]$ and a functor $\phi : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ such that $\iota(\Sigma)$ is a subset of the isomorphisms of $\mathcal{C}[\Sigma^{-1}]$ and such that for any category $\mathcal{A}$ and any functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(\Sigma)$ is a subset of the isomorphisms of $\mathcal{A}$, there exists a unique functor $G : \mathcal{C}[\Sigma^{-1}] \to \mathcal{A}$ such that the following diagram commute :



Lemma 2. Let $\mathcal{C}$ be a category. Let Σ be a set of morphisms of $\mathcal{C}$. Let W be the closure of Σ by composition.

Any localization of $\mathcal{C}$ by Σ is a localization of $\mathcal{C}$ by W. Conversely, any localization of $\mathcal{C}$ by W is a localization of $\mathcal{C}$ by Σ.

Explicit description It is possible to give an explicit description of the localization of a category $\mathcal{C}$. Let us call $\mathcal{G}$ its underlying graph. The set of vertices of $\mathcal{G}$ is the set of objects of $\mathcal{C}$ and the set of edges of $\mathcal{G}$ is the set of morphisms of $\mathcal{C}$. We denote by W the closure by composition of Σ. Let us now call $\mathcal{G}'$ the graph obtained from $\mathcal{G}$ by adding some edges to it : for any $w \in W$, we add $\overline{w}$ to the edges of $\mathcal{G}$. Let $\equiv$ be the smallest relation of equivalence on the morphisms of the category $(\mathcal{G}')^*$ such that :

$$\begin{aligned} w \circ \overline{w} &\equiv id \\ \overline{w} \circ w &\equiv id \\ g \circ f &\equiv g \star f \end{aligned}$$

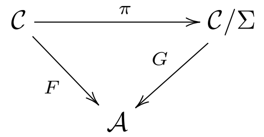
where w is any element of W, $\circ$ is the composition in the category $(\mathcal{G}')^*$ and $\star$ is the composition in $\mathcal{C}$. The localization of $\mathcal{C}$ by Σ is the category $(\mathcal{G}')^*/\equiv$

1.2 Quotient

Definition 3. Let $\mathcal{C}$ be a category. Let Σ be a set of morphisms of $\mathcal{C}$.

A *quotient* of $\mathcal{C}$ by Σ is given by a category $\mathcal{C}/\Sigma$ and a functor $\pi : \mathcal{C} \to \mathcal{C}/\Sigma$ such that $\pi(\Sigma)$ is a subset of the identites of $\mathcal{C}/\Sigma$ and such that for any category

$\mathcal{A}$ and any functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(\Sigma)$ is a subset of the identities of $\mathcal{A}$, there exists a unique functor $G : \mathcal{C}/\Sigma \to \mathcal{A}$ such that the following diagram commute :



Explicit description reference : manu, categories of components and loop-free categories

Given $\mathcal{C}$ a category and Σ a set of morphisms of $\mathcal{C}$, we define the two equivalence relations $\sim_0$ over the set of objects of $\mathcal{C}$ and $\sim_1$ over the set of non-empty $\sim_0$-composable sequences of $\mathcal{C}$ as the smaller equivalence relations satisfying the following conditions :

1. for any $w : x \to y \in \Sigma$, $x \sim_0 y$ and $w \sim_1 id_x \sim_1 id_y$
2. if $x \sim_0 y$, then $(id_x) \sim_1 (id_y)$.
3. if $(\delta_n, ..., \delta_0) \sim_1 (\gamma_p, ..., \gamma_0)$, then the sources of δ_0 and γ_0 are $\sim_0$-equivalent, and the targets of δ_n and γ_p are $\sim_0$-equivalent.
4. if the source of γ is the target of δ (ie $\gamma \circ \delta$ is defined), then $(\gamma, \delta) \sim_1 (\gamma \circ \delta)$.
5. if $(\delta_n, ..., \delta_0) \sim_1 (\delta'_{n'}, ..., \delta'_0)$, $(\gamma_p, ..., \gamma_0) \sim_1 (\gamma'_{p'}, ..., \gamma'_0)$ and the target of δ_n and the source of γ_0 are $\sim_0$-equivalent, then

$$(\gamma_p, ..., \gamma_0, \delta_n, ..., \delta_0) \sim_1 (\gamma'_{p'}, ..., \gamma'_0, \delta'_{n'}, ..., \delta'_0)$$

the quotient of $\mathcal{C}$ by Σ is defined as the category whose objects are the $\sim_0$-classes of equivalence and whose morphisms are the $\sim_1$-classes of equivalence of non-empty $\sim_0$-composable sequences of $\mathcal{C}$.

1.3 Category of fractions

reference : Borceux

Definition 4. Given a category $\mathcal{C}$ and a set Σ of morphisms of $\mathcal{C}$, we say that Σ^* admits a left calculus of fractions when the following conditions hold :

- for $f : A \to B$ in $\mathcal{C}$ and $s : A \to C$ in Σ^* there exist $g : C \to D$ in $\mathcal{C}$ and $t : B \to D$ in Σ^* such that $t \circ f = g \circ s$,
- for $s : A \to B$ in Σ^* and $f, g : B \to C$ in $\mathcal{C}$ such that $f \circ s = g \circ s$, there exist $t : C \to D$ such that $t \circ f = t \circ g$.

Definition 5. Given a category $\mathcal{C}$ and a set Σ of morphisms of $\mathcal{C}$ such that Σ^* admits a left-calculus of fractions in $\mathcal{C}$, we define the category of fraction as the category $\mathcal{D}$ such that

- the objects of $\mathcal{D}$ are the objects of $\mathcal{C}$
- a morphism $A \to B$ in $\mathcal{D}$ is an equivalence class of triples (f, I, s) where :

  - I is an object of $\mathcal{C}$,
  - $f : A \to I$ is a morphism in $\mathcal{C}$,
  - $s : B \to I$ is a morphism in Σ^* and
  - the triples (f, I, s) and (g, J, t) are equivalent if there exist two morphisms x, y in $\mathcal{C}$ such that $x \circ s = y \circ t$ is in Σ^* and such that $x \circ f = y \circ g$

- the composition of the equivalence classes of $(f, I, s) : A \to B$ and $(g, J, t) : B \to C$ in $\mathcal{D}$ is the class of equivalence of $(h \circ f, K, v \circ t) : A \to C$ where $v : J \to K$ is in Σ^*, $h : I \to K$ is in $\mathcal{C}$ and $h \circ s = v \circ g$.

Theorem 6. Given a category $\mathcal{C}$ and a set Σ of morphisms of $\mathcal{C}$ such that Σ^* admits a left-calculus of fractions in $\mathcal{C}$, then the category of fractions is a localization of the category $\mathcal{C}$ by the set of morphisms Σ.

Notations From now on, we call $\mathcal{C}$ a category presented by a 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ and Σ a subset of Σ_1.

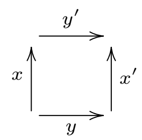
2 Remainings

2.1 Definition of remainings

Hypothesis 7. For any x and y in Σ_1 having same domain, there exists at most one 2-cell $x... \Rightarrow y...$ or $y... \Rightarrow x...$ in Σ_2.

Hypothesis 8. For any x in Σ and any y in Σ_1 having same domain and such that $x \neq y$, there exists a unique x' in Σ^*, a unique y' in Σ_1^* and a unique 2-cell in Σ_2 between xy' and yx'.

Definition 9. For any x in Σ and any y in Σ_1 having same domain and such that $x \neq y$, there exists a unique x' in Σ^*, a unique y' in Σ_1^* and a unique 2-cell in Σ_2 between xy' and yx'. We call x' (resp y') the remaining of x (resp y) after y (resp x) and it is denoted by x/y (resp y/x).



2.2 2-category of remainings

Definition 10. The 2-category of remainings $\mathcal{D}$ is the 2-category generated by the 2-polygraph $(\Sigma_0, \Sigma_1 \uplus \overline{\Sigma}, D_2)$ where

$$\overline{\Sigma} = \{\overline{f} : y \to x \mid f : x \to y \in \Sigma\}$$

and where

$$D_2 = \{\overline{x}y \Rightarrow (y/x)\overline{(x/y)} \mid x \neq y, x \in \Sigma, y \in \Sigma_1\} \uplus \{\overline{x}x \Rightarrow id \mid x \in \Sigma\}$$

Definition 11. We define the preorder $<_1$ on the 1-cells of $\mathcal{D}$ as the smallest preorder such that :

$$\begin{array}{rcl} \overline{x}x >_1 & id & \text{when } x \in \Sigma \\ \overline{x}y >_1 & (y/x)\overline{(x/y)} & \text{when } x \neq y, x \in \Sigma, y \in \Sigma_1 \\ uv_1w >_1 & uv_2w & \text{whenever } v_1 >_1 v_2 \end{array}$$

Hypothesis 12. The preorder $<_1$ has no infinite decreasing sequence.

It means in particular that the rewriting system on the 1-cells of $\mathcal{D}$ and which rewriting rules are given by D_2 is convergent.

Lemma 13. For any 1-cell f of the 2-category of remainings $\mathcal{D}$, there exists unique g in Σ_1^*, h in $\overline{\Sigma}^*$ and A in D_2^* such that $A : f \Rightarrow g\overline{h}$.

Proof. This is done by well-founded induction on the 1-cells of $\mathcal{D}$. Any 1-cell f is of the form

$$f = \overline{a_{1,k_1}}...\overline{a_{1,1}}f_{1,1}...f_{1,j_1}\overline{a_{2,k_2}}...\overline{a_{2,1}}f_{2,1}...f_{2,j_2}...f_{n,1}...f_{n,j_n}$$

faire schéma

There are two cases to consider.

If there does not exist j such that f contains $\overline{a_{j,1}}f_{j,1}$, then it means that f is already in the expected form.

If there exists j such that f contains $\overline{a_{j,1}}f_{j,1}$, then by hypothesis, $\overline{a_{j,1}}f_{j,1}$ rewrites in $(f_{j,1}/a_{j,1})\overline{(a_{j,1}/f_{j,1})}$. Moreover, by definition :

$$\overline{a_{j,1}}f_{j,1} >_1 (f_{j,1}/a_{j,1})\overline{(a_{j,1}/f_{j,1})}$$

which means that f is strictly greater then the term obtained by rewriting $\overline{a_{j,1}}f_{j,1}$.

The uniqueness of $g\overline{h}$ comes from the confluence of the rewriting system on the 1-cells of $\mathcal{D}$ and which rewriting rules are given by D_2.

uniqueness of A : take minimal 1-cell f such that there are 2 possible As. only up to exchange law. □

Lemma 14. For any f in Σ_1^* and γ in Σ^*, there exist unique f/γ in Σ_1^*, a unique γ/f in Σ^* and a 2-cell $f.(\gamma/f) \Rightarrow \gamma.(f/\gamma)$ in $(\Sigma_2 \uplus \overline{\Sigma_2})^*$.

Proof. By the previous lemma, we get unique g in Σ_1^*, h in $\overline{\Sigma}^*$ and A in D_2^* such that $A : f \Rightarrow g\overline{h}$. Besides, by contruction, we get that $h = \gamma/f$ and $g = f/\gamma$.

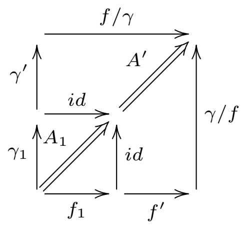
Let us construct from A a 2-cell $\hat{A} : f.(\gamma/f) \Rightarrow \gamma.(f/\gamma)$ in $(\Sigma_2 \uplus \overline{\Sigma_2})^*$ by induction on the size of A (number of generating 2-cells). We may write $\gamma = \gamma' \circ \gamma_1$ where γ_1 is in Σ and $f = f' \circ f_1$ where f_1 is in Σ_1. There are two different cases to consider.

rajouter les identités dans le lemme précédent

First, if $f_1 = \gamma_1$, then by construction of A,

$$A = (\overline{\gamma'}A_1f')A'$$

where $A_1 : \overline{f'}f' \Rightarrow id$:

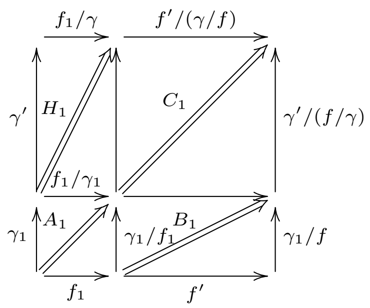


Assuming $\widehat{A'}$ is constructed, we may set

$$\hat{A} = f_1\widehat{A'}.$$

Second, if $f_1 \neq \gamma_1$, then

$$A = \big(\overline{\gamma'}A_1f'\big)\big(H_1\overline{(\gamma_1/f_1)}f'\big)\big(\overline{\gamma'}(f_1/\gamma_1)B_1\big)\big((f_1/\gamma)C_1\overline{(\gamma_1/f)}\big)$$



If there is a 2-cell $\gamma_1.(f_1/\gamma_1) \Rightarrow f_1.(\gamma_1/f_1)$ in Σ_2, we set $\widehat{A_1}$ to be this 2-cell. Otherwise, there is a 2-cell $f_1.(\gamma_1/f_1) \Rightarrow \gamma_1.(f_1/\gamma_1)$ in Σ_2 and we set $\widehat{A_1}$ to be the reverse 2-cell. Assuming $\widehat{H_1}$, $\widehat{B_1}$ and $\widehat{C_1}$ are constructed, we set :

$$\hat{A} = (\gamma_1\widehat{H_1}(f'/(\gamma/f)))(\widehat{A_1}\widehat{C_1})(f_1\widehat{B_1}(\gamma'/(f/\gamma)))$$

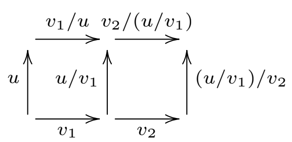
□

Lemma 15. We have extended the definition of remainings to morphisms (and not only generating morphisms). It verifies the following equations. Let u be a morphism in Σ^*.

$$\begin{aligned} id/u &= id \\ u/id &= u. \end{aligned}$$

If either v_1 and v_2 are in Σ^* and u is in Σ_1^*, or v_1 and v_2 are in Σ_1^* and u is in Σ^*, then we may define :

$$\begin{aligned} (v_1.v_2)/u &= (v_1/u).(v_2/(u/v_1)) \\ u/(v_1.v_2) &= (u/v_1)/v_2 \end{aligned}$$



Proof. We have to check that $(u_1.u_2)/(v_1.v_2)$ gives us the same result.

By using the second expression, we get :

$$\begin{array}{rcl} (u_1.u_2)/(v_1.v_2) & = & ((u_1.u_2)/v_1)/v_2 \\ & = & [(u_1/v_1).(u_2/(v_1/u_1))]/v_2 \\ & = & [(u_1/v_1)/v_2].([u_2/(v_1/u_1)]/[v_2/(u_1/v_1)]) \end{array}$$

By using the first expression, we get :

$$\begin{array}{rcl} (u_1.u_2)/(v_1.v_2) & = & [u_1/(v_1.v_2)].(u_2/[(v_1.v_2)/u_1]) \\ & = & [(u_1/v_1)/v_2].(u_2/[(v_1.v_2)/u_1]) \\ & = & [(u_1/v_1)/v_2].(u_2/([v_1/u_1].[v_2/(u_1/v_1)])) \\ & = & [(u_1/v_1)/v_2].([u_2/(v_1/u_1)]/[v_2/(u_1/v_1)]) \end{array}$$

We also have to check that by writing a morphism in two different ways, we still get the same result.

$$\begin{array}{rcl} u/(v_1.(v_2.v_3)) & = & (u/v_1)/(v_2.v_3) \\ & = & ((u/v_1)/v_2)/v_3 \\ & = & (u/(v_1.v_2))/v_3 \\ & = & u/((v_1.v_2).v_3) \end{array}$$

$$\begin{array}{rcl} (v_1.(v_2.v_3))/u & = & (v_1/u).((v_2.v_3)/(u/v_1)) \\ & = & (v_1/u).(v_2/(u/v_1)).(v_3/((u/v_1)/v_2)) \\ & = & ((v_1.v_2)/u).(v_3/((u/v_1)/v_2)) \\ & = & ((v_1.v_2)/u).(v_3/(u/(v_1.v_2)) \\ & = & ((v_1.v_2).v_3)/u \end{array}$$

□

attention, dépend de existence des résidus !

2.3 2-cells of remainings

Here we consider the 2-category $\mathcal{C}'$ generated by the 2-polygraph $(\Sigma_0, \Sigma_1', \Sigma_2')$ where :

$$\Sigma_1' = \{f^H : x \to y \mid f : x \to y \in \Sigma_1\} \cup \{f^V : x \to y \mid f : x \to y \in \Sigma\}$$

and

$$\begin{array}{rcl} \Sigma_2^H & = & \{A_1^H : f^H \to g^H, \\ & & A_2^H : g^H \to f^H \mid A : f \to g \in \Sigma_2\} \\ \Sigma_2^V & = & \{A(f,g)^V : f^V(g/f)^H \to g^H(f/g)^V, \\ & & A(g,f)^V : g^V(f/g)^H \to f^H(g/f)^V \mid f, g \in \Sigma_1, A : f(g/f) \to g(f/g) \in \Sigma_2\} \\ \Sigma_2' & = & \Sigma_2^H \cup \Sigma_2^V \end{array}$$

Given f and g in Σ_1^*, we will denote by $A(f,g)^V$ and $A(g,f)^V$ the corresponding 2-cells between the residuals.

Hypothesis 16. We assume that for any 2-cell $A^H : f^H \to g^H$ in Σ_2^H and any h in Σ_1 such that the residuals between h and f and between h and g exist, then $h/f = h/g$ and there exist a 2-cell $B^H : (f/h)^H \to (g/h)^H$ in $(\Sigma_2^H)^*$.

We define the rewriting system $\mathcal{S}'$ on the 2-cells of $\mathcal{C}'$: for any 2-cell A^H : $f^H \to g^H$ in Σ_2^H and any h in Σ_1 such that the residuals between h and f and between h and g exist,

$$A(h,f)^V.(A^H.(h/f)) \to (h.B^H).A(h,g)^V$$

Lemma 17.

Here we consider the 2-category presented by

3 Cancellativity of the equational morphisms

Lemma 18. If $u \sim u'$ and if v is in W, then

$$v/u = v/u' \quad (1)$$
$$u/v \sim u'/v \quad (2)$$

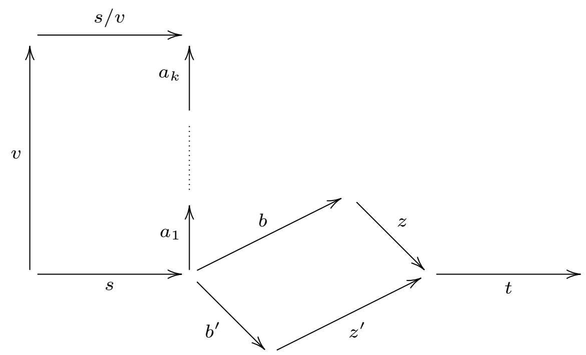
Proof. This is done by induction on the length of the rewriting steps between u and u'.

We have to study the case where $u = s.w.t$ and $u' = s.w'.t$ where (w,w') is in R. By hypothesis, v/s is in W. Let us check that

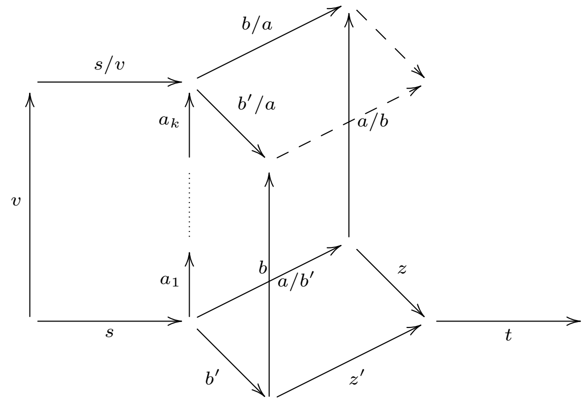
$$w/(v/s) \sim w'/(v/s) \quad \text{and} \quad (v/s)/w = (v/s)/w'.$$

The word v/s writes $a_1.\dots.a_k$ where all a_i are in Σ.

Assume that neither w nor w' are identities. This means in particular that $w = b.z$ and $w' = b'.z'$ with b and b' in X, $z = b'/b$ and $z' = b/b'$ **Pourquoi bordel ?**.

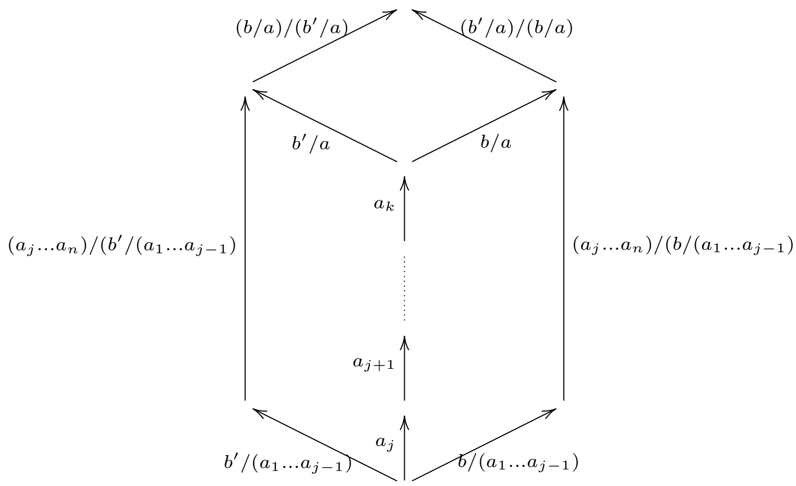


By hypothesis, $a = a_1...a_k$ is in Σ, so we are able to consider its remainings with b and b' :

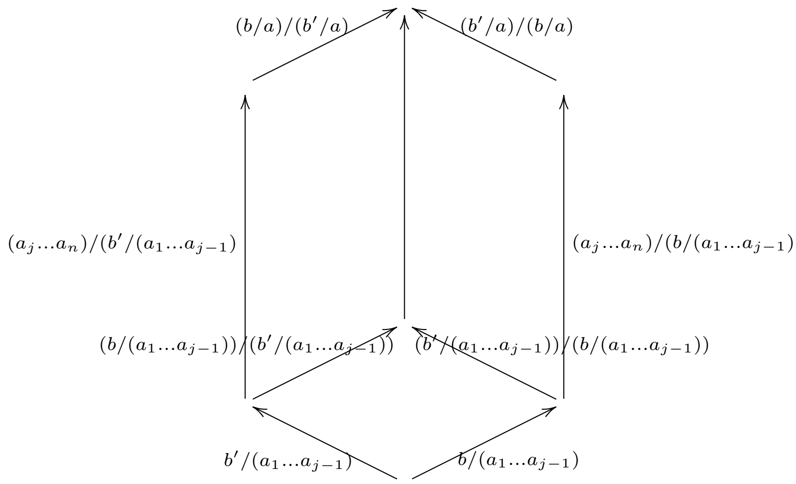


Where the dashed arrows represent $(b'/a)/(b/a)$ and $(b/a)/(b'/a)$.

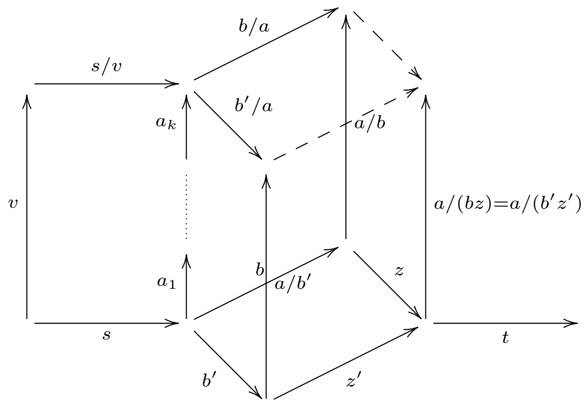
By induction on j from k down to 1, we are able to close the following diagram



into



In particular, we get :



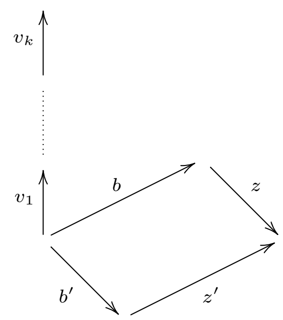
This allows to conclude this proof. □

Lemma 19. If $u \sim u'$ and if u and u' are in W, then

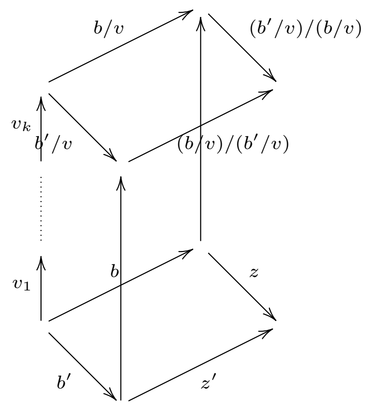
$$v/u = v/u'$$

Proof. This is proven by induction on the length of the path of 2-cells between u and u'. By hypothesis, there exist u_1, ..., u_k in W such that $u_1 = u$, $u_k = u'$ and such that the length of the path between u_i and u_{i-1} is one.

We are then down to considering the following case, where $(bz, b'z')$ is a relation in R with bz and $b'z'$ in W.



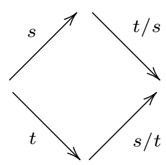
This means in particular that we are able to construct the remainings between b and $v = v_1...v_k$ and between b' and v. Besides, both b/v and b'/v are in W which means that we are able to consider their remainings. This gives us the following diagram



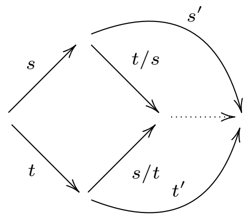
Similarly to previous lemma, we are able to **ce serait probablement une bonne idée de le faire quand meme, quitte à recopier et adapter un pouième** fill this diagram using the cube property. ☐

Lemma 20. The category $\mathcal{C}$ admits pushouts between equational morphisms.

Proof. Let us consider two morsphisms s and t of W. By hypothesis, there exist remainings between them, which gives :



Let us now verify that the remainings satisfy the universal property of pushouts. Let us consider the following case :



There are two possible dotted arrows : $(ss')/(s.(t/s))$ and $(tt')/(t.(s/t))$. Indeed,

$$\begin{array}{rcll} (s.(t/s))/(ss') & = & (t/s)/s' & \\ & = & t/(ss') & \\ & = & t/(tt') & \text{1 du premier lemme préparatoire} \\ & = & 1 & \end{array}$$

Similarly $(t.(s/t))/(tt') = 1$.

Let us now prove that these two arrows are the same in the presented category, which means that

$$(ss')/(s.(t/s)) \sim (tt')/(t.(s/t)).$$

$$\begin{array}{rcll} (ss')/(s.(t/s)) & = & (ss')/(t.(s/t)) & \text{2e lemme préparatoire} \\ & \sim & (tt')/(t.(s/t)) & \end{array}$$

Let us assume that there is another possible h. **et là, on fait quoi ?** □

faire attention que dans C, les morphismes équationnels sont en fait dans Σ^*/Σ_2

Lemma 21. Whenever

$$f \circ s = g \circ s$$

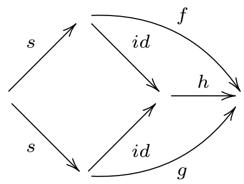
for f and g two morphisms of $\mathcal{C}$ and s a morphism in W, then

$$f = g$$

Proof. Let us assume that

$$f \circ s = g \circ s$$

for f and g two morphisms of $\mathcal{C}$ and s a morphism in W. As there are pushouts between morphisms of W, we get that there exists a unique h in $\mathcal{C}$ such that



In particular, we get that $f = g = h$ in $\mathcal{C}$

□

Lemma 22. The category $\mathcal{C}$ admits pushouts between equational morphisms and other morphisms.

Proof. Let us consider an equational morphism f and a morphism g having same domain. By a previous lemma, we may consider their remainings. Let us now check that this is indeed a pushout. Let f' and g' be two morphisms such that in $\mathcal{C}$,

$$f' \circ f = g' \circ g.$$

The remaining f/g is equational, which means that we may consider its remainings with g' :

$$(f/g)/g' = f/(gg') = f/(ff') = id$$

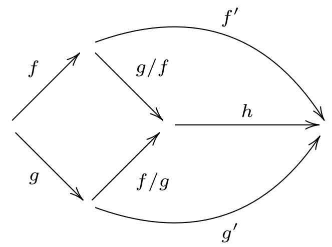
here we assume $u \sim v$ **and f eq than** $f/u \sim f/v$ Let us denote $k = g'/(f/g)$.

Which means that $g' = k \circ (f/g)$. Let us prove that $f' = k \circ (g/f)$.

$$\begin{aligned} ff' &= gg' \\ &= g.(f/g).k \\ &= f.(g/f).k \end{aligned}$$

And by cancellativity of the equational morphisms we get the expected result.

Now, let us assume that there exist a morphism h making the following diagrams commute :



As f/g is an equational morphism, we get directly from $(f/g).h = f' = (f/g).k$ that $h = k$. □

Definition 23. The local property of the cylinder is satisfied if for any two 1-cells f and g in Σ_1 with a 2-cell $u_1 \Rightarrow u_2$ or $u_2 \Rightarrow u_1$ in Σ_2, for any 1-cell h in Σ (having the same domain), it is possible to take their remainings and moreover

$$h/f = h/g$$

and there is a 2-cell $f/h \Rightarrow g/h$ in $(\Sigma_2 \uplus \overline{\Sigma_2})^*$.

4 Category of normal forms

Let us consider the rewriting system on the objects of $\mathcal{C}$, namely Σ_0, and which relations are the arrows of Σ. We ask that this rewriting system is convergent (terminating and locally confluent). This means in particular that every object x of $\mathcal{C}$ admits a unique normal form with respect to Σ denoted $\hat{x}$.

Definition 24. We define the category of normal forms, denoted $\mathcal{C}_\Sigma$ as the full subcategory of $\mathcal{C}$ whose objects are the normal forms of the rewriting system (Σ_0, Σ).

Lemma 25. The inclusion functor $\iota : \mathcal{C}_\Sigma \to \mathcal{C}$ is full and faithful.

Proof. The objects of $\mathcal{C}_\Sigma$ are objects of $\mathcal{C}$. Let x and y be two objects of $\mathcal{C}_\Sigma$. As $\mathcal{C}_\Sigma$ is a full subcategory of $\mathcal{C}$,

$$Hom_{\mathcal{C}_\Sigma}(x, y) = Hom_{\mathcal{C}}(x, y) = Hom_{\mathcal{C}}(\iota x, \iota y)$$

This proves that the functor ι is full and faithful. □

The functor ι does not define an equivalence of category as the morphisms in Σ are not isomorphisms in $\mathcal{C}$.

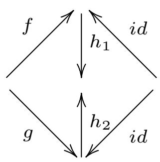
5 Embedding of $\mathcal{C}$ into its localization

Lemma 26. Let $\mathcal{C}$ be a category and let W be a set of morphisms of $\mathcal{C}$. Consider the corresponding category of fractions $\phi : \mathcal{C} \to \mathcal{C}[W^{-1}]$.

If W admits a left calculus of fractions and all the morphisms of W are monomorphisms, then ϕ is faithful.

Proof. Let f and g be two morphisms of $\mathcal{C}$ such that $\phi f = \phi g$. Let us show that $f = g$.

Given the explicit description of the category of fractions of prop 5.2.4 of Borceux, we know that ϕf is the equivalence class of (f, id). As $\phi f = \phi g$, there exist h_1 and h_2 such that $h_1 \circ f = h_2 \circ g$ and $h_1 \circ id = h_2 \circ id \in W$:



We get that $h_1 = h_2 = h \in W$, and thus $h \circ f = h \circ g$. Besides, all the morphisms of W are monomorphisms, which means that $f = g$.

□

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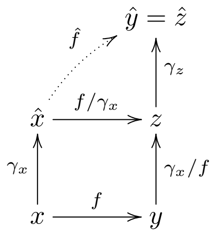
6 Isomorphism between the quotient and the category of normal forms

Lemma 27. The category of normal forms $\mathcal{C}_\Sigma$ is a quotient of the category $\mathcal{C}$ by Σ.

Proof. First let us define the functor π between $\mathcal{C}$ and $\mathcal{C}_\Sigma$:

$$\begin{array}{rcl} \pi : \mathcal{C} & \to & \mathcal{C}_\Sigma \\ x & \mapsto & \hat{x} \\ (f : x \to y) & \mapsto & (\hat{f} : \hat{x} \to \hat{y}) \end{array}$$

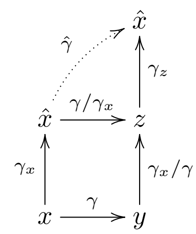
where $\hat{f}$ is defined (using the lemma on the remainings of non-generating morphisms) as :



where $\gamma_x : x \to \hat{x}$ and $\gamma_z : z \to \hat{z}$ are in Σ^*.

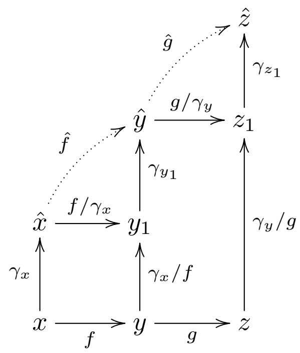
By cancellativity of the morphisms in Σ^*, we get that $\hat{f}$ is defined independantly of the choice of morphisms γ_x and γ_z.

Let us check that $\hat{\gamma} = id$ for γ in Σ^*. By convergence of the rewriting system (Σ_0, Σ), $\hat{x} = \hat{y} = \hat{z}$, which means that

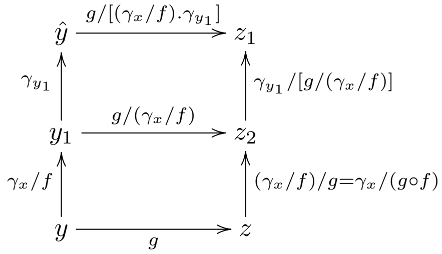


As $\hat{x}$ is in normal form, there is no non-identity morphism in Σ^* which domain is $\hat{x}$. This means in particular that $\gamma/\gamma_x = id$, $z = \hat{x}$ and $\gamma_z = id$. This implies that $\hat{\gamma} = id$.

Let us check that $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.



where $\gamma_y = \gamma_{y_1} \circ (\gamma_x/f)$. Moreover, the down-right rectangle is



Which gives us that $\widehat{g \circ f} = \hat{g} \circ \hat{f}$ as

$$(g \circ f)/\gamma_x = (g/(\gamma_x/f)) \circ (f/\gamma_x)$$

We have

$$\pi \circ \iota = id.$$

Let us now consider a functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(\Sigma)$ is a subset of the identities of $\mathcal{A}$. Let us define $G = F\iota$. Let us check that $F = G\pi$. Let x be an object of $\mathcal{C}$ and let $\gamma : x \to \hat{x}$ a morphism of $\mathcal{C}$ in Σ^*. We get that

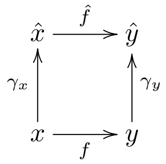
$$Fx = F\gamma x = F\hat{x}$$

as $F(\Sigma)$ is a subset of the identities of $\mathcal{A}$. Moreover,

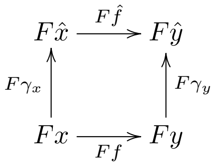
$$(G \circ \pi)x = G\hat{x} = (F\iota)\hat{x} = F\hat{x}.$$

This proves the expected equality on the objects of $\mathcal{C}$.

Let us now consider a morphism $f : x \to y$ in $\mathcal{C}$. In $\mathcal{C}$, we have the following commutating diagram :



with γ_x and γ_y two morphisms in Σ^*. By functoriality of F, we get the following commutating diagram in $\mathcal{A}$:



where $F\gamma_x = id$ and $F\gamma_y = id$, which means that

$$Ff = F\hat{f}$$

Besides in $\mathcal{C}$, $\iota\pi f = \hat{f}$, which gives

$$F\hat{f} = F\iota\pi f = G\pi f.$$

This proves the expected equality on the morphisms of $\mathcal{C}$.

There remains to show that G is the unique functor that allows to check the universal property of the quotient. Let us assume that both G_1 and G_2 satisfy the universal property. This means that $G_1\pi = F = G_2\pi$. Using $\pi\iota = id$, we get :

$$G_1 = F\iota = G_2$$

which proves uniqueness of G. □

Corollary 28. The categories $\mathcal{C}_\Sigma$ and $\mathcal{C}/\Sigma$ are isomorphic.

7 Equivalence between the category of normal forms and the localization

Lemma 29. The categories $\mathcal{C}_N$ and $\mathcal{C}[\Sigma^{-1}]$ are equivalent.

Definition 30. Given two categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $S : \mathcal{C} \to \mathcal{D}$, the functor S defines an equivalence of category if the following conditions are satisfied :

1. for any object d of $\mathcal{D}$, there exists an object c of $\mathcal{C}$ such that d and Sc are isomorphic,

2. the functor S is both full and faithful, namely, for any two objects c_1 and c_2 of $\mathcal{C}$, the map $Hom_{\mathcal{C}}(c_1, c_2) \to Hom_{\mathcal{D}}(Fc_1, Fc_2)$ induced by S is bijective.

Proof. There is an inclusion functor $\iota : \mathcal{C}_\Sigma \to \mathcal{C}$ and there is also a localization functor $\phi : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$.

They allow us to define the functor $S : \mathcal{C}_\Sigma \to \mathcal{C}[\Sigma^{-1}]$ as the composition

$$\mathcal{C}_\Sigma \xrightarrow{\iota} \mathcal{C} \xrightarrow{\phi} \mathcal{C}[\Sigma^{-1}]$$

We now aim at showing that this functor defines an equivalence of category.

We have previously shown that the functor ϕ is faithful. The functor ι is also faithful. This means that the functor S is faithful.

Let us now prove that it is full, namely that the function

$$\mathcal{C}_\Sigma(\hat{x}, \hat{y}) \to \mathcal{C}[\Sigma^{-1}](\hat{x}, \hat{y})$$

induced by S is surjective. Let $f : \hat{x} \to \hat{y}$ be a morphism in $\mathcal{C}[\Sigma^{-1}]$. By a previous lemma, there exist g in Σ_1^* and h in Σ^* such that $f = g\overline{h}$ in $\mathcal{C}[\Sigma^{-1}]$. Besides, as $\hat{y}$ is in normal form, $h = id$. This means that $Sg = g$ and therefore S is full.

Any object y of $\mathcal{C}[\Sigma^{-1}]$ is also an object of $\mathcal{C}$ and $\phi y = y$. We may consider its normal form $\hat{y}$ which is an object of $\mathcal{C}_N$. By definition, there is a morphisms $h : y \to \hat{y}$ in Σ^*. The morphism ϕh is an isomorphism in $\mathcal{C}[\Sigma^{-1}]$ which means that the objects y and $S\hat{y}$ are isomorphic.

□

8 Extending this to 2-categories